65. On Regular Algebraic Systems

A Note on Notes by Iseki, Kovacs, and Lajos

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L. Kovacs [2], K. Iseki [1], and S. Lajos [3] characterized regular rings and semigroups as algebraic systems satisfying the property $R\cap L=RL$ for any right ideal R and any left ideal L. A semigroup $(S,\cdot)$ and a ring or semiring $(S,+,\cdot)$ is regular iff for each $s\in S$ there exists an $x\in S$ such that $sxs=s$. Clearly, this follows from the statement: for each $s\in S$, there exist $x, y\in S$ such that $sxys=s$. The two statements are equivalent, for, if for each $s\in S$ there exists an $x\in S$ such that $sxs=s$, then also there exist a $z\in S$ such that $x=xzx=x(zx)=xy$ and therefore $sxys=s$.

In this communication we shall give a unified generalization of the characterizations of Kovacs, Iseki, and Lajos. It turns out that the description of regularity in terms of ideals is intrinsic to associative operations in general.

By an *algebraic system* $(A, o_1,\cdots, o_n)$ or simply A is meant a set A closed under a collection of m_i-ary operations o_i and often also satisfying a fixed set of laws. For instance, an m-ary operation $(\cdots)$ on A satisfies the *associative law* iff for each $x_1,\cdots,x_{2m-1}\in A$, $((x_1x_2\cdots x_m)x_{m-1}\cdots x_{2m-1})=(x_1(x_2x_3\cdots x_{m-1})\cdots x_{2m-1})=\cdots=(x_1x_2\cdots(x_{m-1}x_{m-2}\cdots x_{2m-1}))$. A is said to be *regular* with respect to the operation $(\cdots)$ iff for each $a\in A$ there exist $x_2, x_3,\cdots, x_m; y_1\, y_3,\cdots, y_m;\cdots; z_1, z_2,\cdots, z_{m-1}\in A$ such that

$$((ax_2\cdots x_m)(y_1ay_3\cdots y_m)\cdots(z_1z_2\cdots z_{m-1}a))=a.$$

Note that if A is both associative and regular relative to the operation, then the preceding identity may be rewritten as follows:

$$((ax_2\cdots x_m)(y_1a\cdots y_m)\cdots(z_1z_2\cdots a))=(a(x_2\cdots x_my_1)a\cdots(z_1z_2\cdots z_{m-1})a))$$
$$=(av_1a\cdots(\cdots v_{m-1}a))=a \text{ for some } v_1,\cdots,v_{m-1}\in A.$$

A subset S of A constitues a *subsystem* iff S is closed under the same operations and satisfies the same fixed laws in A.

Following G. B. Preston [4], a *j-ideal* $j=1,\cdots,m$ relative to the m-ary operation $(\cdots)$ is defined to be a subsystem I_j such that for any $x_1, x_2,\cdots, x_m\in A$, if $x_j\in I_j$ then $(x_1x_2\cdots x_m)\in I_j$. The j-ideal relative to $(\cdots)$ generated by an element $a\in A$ (usually called a principal j-ideal) is denoted by

$$(a)_j=(AA\cdots\overset{j}{a}\cdots A)\cup\{a\}.$$