64. On a Special Metric Characterizing a Metric Space of dim $\leqq n$

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(Comm. by Kinjirô KUNUGI, M.J.A., May 11, 1963)

Once we have characterized [3] a metric space of covering dimension $\leqq n$ by means of a special metric as follows.

A metric space R has dim$\leqq n$ if and only if we can introduce a metric ρ in R which satisfies the following condition: For every $\varepsilon>0$ and for every $n+3$ points $x, y_1, \cdots, y_{n+2}$ in R satisfying[1)]

$$\rho(S_{\varepsilon/2}(x), y_i)<\varepsilon, \quad i=1,\cdots,n+2,$$

there is a pair of indices i, j such that

$$\rho(y_i, y_j)<\varepsilon \quad (i\neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows.

A separable metric space R has dim$\leqq n$ if and only if we can introduce a totally bounded metric ρ in R which satisfies the following condition:

For every $n+3$ points $x, y_1, \cdots, y_{n+2}$ in R, there is a triplet of indices, i, j, k such that

$$\rho(y_i, y_j)\leqq\rho(x, y_k) \quad (i\neq j).$$

The first theorem is not so smart though it is valid for every metric space. The problem of generalizing the second theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

Theorem. *A metric space R has dim$\leqq n$ if and only if we can introduce a metric ρ into R which satifises the following condition:*

For every $n+3$ points $x, y_1, \cdots y_{n+2}$ in R, there is a pair of indices, i, j such that

$$\rho(y_i, y_j)\leqq\rho(x, y_j) \quad (i\neq j).$$

Proof. The proof of this theorem is never simple.[2)] Here we shall only show the proof of sufficiency and the outline of the proof of necessity.

Sufficiency. We shall prove that the following weaker condition is sufficient for R to have dim $\leqq n$.

We can introduce a metric ρ into R such that for a definite

1) $S_{\varepsilon/2}(x)=\{y \mid \rho(x, y)<\varepsilon/2\}$.

2) The detailed proof will be published in some other place.