60. *A Remark on Gentzen's Paper "Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie". I*

By Gaisi TAKEUTI

Department of Mathematics, Tokyo University of Education, Tokyo

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In his paper [2], G. Gentzen proved the following theorem:

The transfinite induction up to the first ε-number ε_0 is not provable in the theory of natural numbers (formalized in the first order predicate calculus), while the transfinite induction up to an arbitrary ordinal number less than ε_0 is provable in the theory of natural numbers.

For the proof he introduced ordinal numbers up to ε_0 as individual constants of the system. But, following his method, the theorem can be stated in a more general form. That is, the linear ordering which is to be proved as a well-ordering in the system may be any general recursive linear ordering. The purpose of this paper is to remark this fact and state the following theorem in several cases. We shall begin with defining some notions and notations.

Let S be a (constructively defined) set which is well-ordered by a relation $<^*$. For any element s of S, $|s|$ stands for the order-type represented by s in the sense of $<^*$. We shall call $|s|$ the *value* of s in the sense of $<^*$. By $|S|$ we shall denote the least ordinal number α such that $|s|<\alpha$ for every $s \in S$. $|S|$ is called the *value* of S.

Let $\mathfrak{S}$ be a theory of natural numbers (formalized in the first or second order predicate calculus of Gentzen's style (cf. [1])). Throughout this paper we allow every sequence $\Gamma \rightarrow \Delta$ with the following properties as a mathematische Grundsequenz; every formula consisting of Γ or Δ is general recursive and containing no logical symbol and every sequence obtained from $\Gamma \rightarrow \Delta$ by replacing all free variables in Γ, Δ by arbitrary numerals (terms of the form $0'^{\cdots\prime}$) is true. The formal consistency of every consistent axiomatizable system can be proved in $\mathfrak{S}$. For this reason our result of this paper does not follow from Gödel's incompleteness theorem.

Let $P(a)$ be a general recursive predicate (containing no logical symbol) and $a \prec b$ be a linear ordering of the set $\{a \mid P(a)\}$. If it is provable in $\mathfrak{S}$ that $a \prec b$ is a well-ordering of the set, we call $a \prec b$ a *provable well-ordering* in $\mathfrak{S}$.

Theorem. *Let $\mathfrak{S}$ be a theory of natural numbers and S a con-*