79. A Characteristic Property of L_ρ-Spaces ($\rho \geqq 1$). III

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The aim of this paper is to give a characterization of the abstract L_ρ-space[1] ($\rho \geqq 1$) in terms of the norm.

Through this paper, let $\boldsymbol{R}$ be a Banach lattice with a continuous semi-order.[2]

$\boldsymbol{R}$ is called the abstract L_ρ-space, if the norm satisfies the following condition:

(L_ρ) $\quad \|x+y\|^\rho = \|x\|^\rho + \|y\|^\rho$ for every $|x| \frown |y| = 0$, $x, y \in \boldsymbol{R}$.

When we consider the case which the norm has the restricted Gateaux's differential i.e.,

(RG) $$G(x; [p]x) = \lim_{\varepsilon \to 0} \frac{\|x+\varepsilon[p]x\| - \|x\|}{\varepsilon}$$

exists for each $\|x\| \leqq 1$ and each projector $[p]$,[3] it is easily seen that for numbers α, β and projectors $[p], [q]$

(1) $$G(x; \alpha[p]x + \beta[p]y) = \alpha G(x; [p]x) + \beta G(x; [q]x)$$

if the right side has a sense.

Used the condition (RG), our characterization is described in the following form.

Theorem. *Suppose that $\boldsymbol{R}$ is at least three dimensional space. In order that $\boldsymbol{R}$ is the abstract L_ρ-space for some $\rho \geqq 1$, it is necessary and sufficient that the norm on $\boldsymbol{R}$ satisfies the conditions* (RG) *and*

(*) $$G(a+x; a) = G(a+y; a)$$

for every $a \frown x = a \frown y = 0$ and $\|a+x\| = \|a+y\| = 1$.

Remark. It is known that the Gateaux's differential produces the equality in the Hölder's inequality. In this sense, our theorem is closely related to the previous paper [4 and 5], especially, if the conjugately similar transformation $\boldsymbol{T}$ preserves the norm then $\|a+x\| = \|a+y\| = 1$ and $a \frown x = a \frown y = 0$ imply

$$G(a+x; a) = \frac{(a, \boldsymbol{T}(a+x))}{\|\boldsymbol{T}(a+x)\|} = \frac{(a, \boldsymbol{T}a)}{\|\boldsymbol{T}(a+x)\|} = \frac{(a, \boldsymbol{T}(a+y))}{\|\boldsymbol{T}(a+y)\|} = G(a+y; a)$$

because for $\|x\| = 1$ we have $(x, \boldsymbol{T}x) = \|\boldsymbol{T}x\|$ and hence $G(x; [p]x)$

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1) See [3: p. 312]. The braquet [·] denotes the number of the reference in the last.

2) A semi-order is said to be *continuous*, if for any $x_\nu \downarrow_{\nu=1}^\infty$ and $0 \leqq x_\nu \in \boldsymbol{R}$ there exists x such that $x_\nu \downarrow_{\nu=1}^\infty x$.

3) For any $p \in \boldsymbol{R}$, $[p]x = \bigcup_{n=1}^\infty (|p| \frown nx^+) - \bigcup_{n=1}^\infty (|p| \frown nx^-)$ where $x^+ = x \smile 0$ and $x^- = (-x)^+$.