78. *On the Uniqueness of Solutions of the Cauchy Problem for Hypoelliptic Partial Differential Operators*

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1. Introduction. L. Härmander in the note [2] proved the following: For a differential operator L with constant coefficients the uniqueness of solutions for the Cauchy problem does not hold in the class C^∞, if the initial plane is characteristic for L.

The object of this note is to prove the uniqueness of solutions for the Cauchy problem, whose initial plane may be characteristic, under the restriction of the class of possible solutions which belong to $\{u; \exp(-(1+|x|^2)^{1/2})\cdot u \in C^m_{[0,T]}(\mathfrak{H}_{2m})\}$ defined in the section 2. Let $R^{1+\nu}$ be the $(1+\nu)$-dimensional Euclidean space with coordinates $(t,x)=(t, x_1,\cdots,x_\nu)$, and $(m,\mathfrak{m})=(m, m_1,\cdots,m_\nu)$ $(m_j\leqq 2m;\ j=1,\cdots,\nu)$ be an appropriate real vector whose elements are positive integers. We shall consider differential operators L of the form

$$L=\sum_{i/m+|\alpha:\mathfrak{m}|=1} a_{i,\alpha}(t,x)\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}\quad (a_{m,0}(t,x)=1) \tag{1.1}$$

$$(\alpha=(\alpha_1,\cdots,\alpha_\nu),\ x^\alpha=x_1^{\alpha_1}\cdots x_\nu^{\alpha_\nu},$$
$$|\alpha|=\alpha_1+\cdots\alpha_\nu,\ |\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_\nu/m_\nu)$$

where $a_{i,\alpha}(t,x)$ belong to $\mathscr{B}_{(t,x)}$ in $[0,T]\times R^\nu$. Here we remark this class is an extension of the result of S. Mizohata [6] and contains the operators of the form

$$L=(-1)^{[s/2]}\frac{\partial^s}{\partial t^s}+(-1)^m\sum_{|\alpha|=2m}A_\alpha(t,x)\frac{\partial^{2m}}{\partial x^\alpha}\ {}^{1)} \tag{1.2}$$

where $s\geqq m$ and $\sum_{|\alpha|=2m}A_\alpha(t,x)\xi^\alpha\geqq\delta>0$ for $|\xi|=1$; see [5].

2. Definitions and lemmas. We set the associated polynomials $L(t,x,\lambda,\xi)$ of (1.1) for real vectors $(\lambda,\xi)=(\lambda,\xi_1,\cdots,\xi_\nu)$

$$L(t,x,\lambda,\xi)=\sum_{i/m+|\alpha:\mathfrak{m}|=1}a_{i,\alpha}(t,x)\lambda^i\xi^\alpha\ \ (a_{m,0}(t,x)=1). \tag{2.1}$$

Let us define $r=r(\xi)$ as a positive root of the equation $\sum_{j=1}^{\nu}\xi_j^2 r^{-2/m_j}=1$ $(\xi\neq 0)$, and set $K(\xi)=\{\sum_{j=1}^{\nu}\xi_j^{2m_j}\}^{1/4m}$. Then, we have

$$\nu^{1/4m}K(\xi)\leqq r^{1/2m}\leqq\nu^{1/2}K(\xi),\ |\partial^{|\alpha|}/\partial\xi^\alpha r(\xi)^{1/2m}|\leqq C_\alpha{}^{2)}K(\xi)^{1-2m|\alpha:\mathfrak{m}|}. \tag{2.2}$$

The proof is given in [4], but in this case we must replace m by $2m$.

We denote by $\mathfrak{H}_p$ a function space

$$\mathfrak{H}_p=\left\{u\in L^2(R^\nu);\ \|u\|_p^2=\int(1+K(\xi))^{2p}|\hat{u}(\xi)|^2d\xi<\infty\right\},$$

1) For these operators, $a_i(t)$ in (3.1) of this note are constants.
2) In what follows constants C are always positive.