76. *Note on the Modular Forms*

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1. In his paper [2] Bochner treated the modular forms of level 1. We shall add a little to his result. In the following we shall use freely the notaions and the results in the papers of Bochner and of ourselves [5].

2. By the theory of Bochner we have

Theorem of Bochner. *Let λ and k be positive numbers and $f(z)$ be an analytic function defined on the upper half plane such that $f(z+\lambda)=f(z)$ and $f(z)=\pm\left(\frac{i}{z}\right)^k f\left(-\frac{1}{z}\right)$. Let $\sum_{n=0}^{\infty} a_n e^{\frac{2\pi}{\lambda}nzi}$ be the Fourier series of $f(z)$ and $\sum_{n=0} a_n n^{-s}$ be convergent for some s. Then*

$$\sum_{n=0}^{\infty} a_n \varphi(\sqrt{n}) = \pm \sum_{n=0}^{\infty} a_n T_{\lambda,2k}\varphi(\sqrt{n})$$

for any φ in $\mathfrak{P}_0$, where $T_{\lambda,2k}\varphi$ is the Bochner transform of φ.

From now we shall consider the case where $\lambda=1, k$ is an even number, $a_0=0$ and $f(z)=z^{-k}f\left(-\frac{1}{z}\right)$. In this case $\sum_{n=1}^{\infty} a_n e^{2\pi nzi}$ is a cusp form of dimension $-k$ and of level 1. By the general theory of cusp form (Hecke [4] p. 652) we know $\sum_{n=1} \frac{a_n}{n^s}$ converges absolutely for $\operatorname{Re} s > \frac{k+1}{2}$. Using the above theorem of Bochner we can prove

Proposition 1. *Let k be an even natural number and $\sum_{n=1}^{\infty} a_n e^{2\pi nxi}$ be a cusp form of dimension $-k$ and of level* 1. *If $f(x)$ is a function of class C^∞ such that $\sum_{n=1}^{\infty} a_n f(\sqrt{n})$ is convergent and $\int_0^\infty x^{k+\frac{3}{2}} \left| \left(\frac{d}{xdx}\right)^2 f(x) \right| dx$ exists, then*

$$\sum_{n=1}^{\infty} a_n f(\sqrt{n}) = (-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} a_n T_{1,2k} f(\sqrt{n}).$$

Proof. We have $|T_{1,2k}f(\sqrt{n})| = O(n^{-\frac{k}{2}-\frac{3}{4}})$ by Proposition 4 in [5]. Therefore $\sum_{n=1}^{\infty} a_n Tf(\sqrt{n})$ is absolutely convergent by Hecke's theorem.

Now we take functions $\varphi_1(x), \varphi_2(x), \cdots$ in $\mathfrak{P}_0$ such that

$$\begin{aligned} &\varphi_m(x) = f(x) && \text{for } 0 \le x \le \sqrt{m}, \\ &\varphi_m(x) = 0 && \text{for } x \ge \sqrt{m+1} \text{ and} \\ &|\varphi_m(x)| \le |f(\sqrt{m})| && \text{for } \sqrt{m} < x < \sqrt{m+1}. \end{aligned}$$