75. A Generalization of König's Lemma

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(Comm. by Zyoiti SUETUNA, M.J.A., June 12, 1963)

König proved in [1] the following lemma:

Let $E_1, E_2, \cdots, E_n, \cdots$ *be an enumerable sequence of finite and non empty sets and* R *a relation of two arguments satisfying the following condition: for every element* x_{n+1} *of* E_{n+1} $(n \geq 0)$ *there is an element* x_n *of* E_n *corresponding to* x_{n+1} *by the relation* R i.e. $x_n R x_{n+1}$. *Then we can obtain an infinite sequence* $a_1, a_2, \cdots, a_n, \cdots$ *such that* $a_n \in E_n$ *and* $a_n R a_{n+1}$ $(n=1, 2, \cdots)$.

Sometimes, it is also called Brouwer's fan theorem.

In this paper we shall prove a generalization of this lemma. Let R be a set. p stands for the element of R. A finite set W_p is assigned for every $p \in R$. If R_1 is a subset of R and f is an element of $\prod_{p \in R_1} W_p$, then f is called a *partial function* (over R) and $D(f)$ (the domain of f) is defined to be R_1. If $D(f)=R$, then f is called a *total function*. If f and g are partial functions and $D(f)=D_0 \subseteq D(g)$ and $f(x)=g(x)$ for every $x \in D_0$, then we call g an *extension* of f and write $f \prec g$ and $f = g \restriction D_0$. If $f \prec g$ and $D(g)=D(f) \cup N$, then we say 'g is an extension of f over N'.

THEOREM. *Let* P *be a property about partial function satisfying the following conditions*:

1. $P(f)$ *holds if and only if there exists a finite subset* N *of* R *satisfying* $P(f \restriction N)$.
2. $P(f)$ *holds for every total function* f.

Then there exists a finite subset N_0 *of* R *such that* $P(f)$ *holds if* $D(f) \supseteq N_0$.

It is noted that $\overline{\overline{R}}$ be arbitrary large. The case that R is the set of natural numbers is the original König's lemma.

To prove this theorem we shall first define several concepts. We say $\widetilde{P}(f)$ if there exists a finite subset N of R such that every extension of f over N satisfies P. Clearly $\widetilde{P}(f)$ holds for every total function.

We define $f * g$ to be the function uniquely defined by the following conditions:

1) $D(f * g) = D(f) \cup D(g)$
2) $f \prec f * g$
3) If $p \in D(g) - D(f)$, then $(f * g)(p) = g(p)$.