# 102. Cyclic and Homogenous m-Semigroups

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In a previous note [6], the author has indicated how a number of results in semigroups can be extended to more general algebraic systems consisting of an arbitrary associative $m$-ary operation. These latter systems may be called *m-semigroups*. Ordinary semigroups are thus 2-semigroups. A corresponding theory of *m-groups* has been in existence for quite some time (see W. Dörnte [1] and E. L. Post [4]).

In the present communication, we shall pursue further this trend of generalization in the particular topic mentioned in the title. The reader is referred to the previous paper [6] for other pertinent notions and definitions.

For any $m$-semigroup $A$, the subsystem $[a]$ generated by an element $a\in A$ consists of all admissible powers of $a$:

$$a=a^{\langle 0\rangle},\ a^m=a^{\langle 1\rangle},\cdots,\ a^{k(m-1)+1}=a^{\langle k\rangle},\cdots.$$

Two instances are possible:

I. No pair of admissible powers of $a$ are equal so that $[a]$ is countably infinite;

II. There exists two non-negative integers $r$ and $s$ with $r<s$ such that $a^{\langle r\rangle}=a^{\langle s\rangle}$. Without loss of generality $s$ may be assumed to be the least possible such integer. Let $p=s-r$ so that $a^{\langle r\rangle}=a^{\langle r+p\rangle}$. Then by induction $a^{\langle r\rangle}=a^{\langle r+kp\rangle}$ for all integers $k\geqq 0$. On the other hand, for any non-negative integer $n$, one has $n=kp+i$, where $k\geqq 0$ and $0\leqq i<p$. Hence

$$a^{\langle r+n\rangle}=a^{\langle r+(kp+i)\rangle}=a^{\langle r+i\rangle}.$$

This means that every admissible power of $a$ beyond the $\langle s-1\rangle$th is an element of the set

$$G_a=\{a^{\langle r\rangle},\ a^{\langle r+1\rangle},\cdots,\ a^{\langle s-1\rangle}\}.$$

Note that $a^{\langle x\rangle}=a^{\langle y\rangle}$ if and only if $x\equiv y \pmod{p(m-1)}$. The order of $[a]$ is thus $s=r+p$, where $p$ is the order of $G_a$ (or the *period* of $a$) and $r$ is the *index* of $a$. $A$ is said to be *cyclic* if and only if $A=[a]$ for some $a\in A$.

Note further that $G_a$ is closed under the same $m$-ary operation in $A$ and is therefore an $m$-subsemigroup of $A$. That it is an ideal of $[a]$ is evident. $G_a$ is in fact a minimal ideal, for, if $x\in G_a$ and $x_i$ belongs to any ideal $I\subseteqq G_a$, then by a property of $m$-groups there exist $m-1$ elements $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_m\in G_a$ such that $(x_1\cdots x_{i-1}x_i x_{i+1}\cdots x_m)=x$. Thus $x\in I$ and therefore $G_a=I$. The maximality of