101. Nörlund Summability of a Sequence of Fourier Coefficients

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1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots p_n.$$

The sequence to sequence transformation, viz.

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} s_\nu}{P_n} = \sum_{\nu=0}^{n} \frac{p_\nu s_{n-\nu}}{P_n}, \quad (P_n \neq 0), \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of constants $\{p_n\}$. The series $\sum a_n$ or the sequence $\{s_n\}$ is said to be summable by Nörlund means, or summable (N, p_n) to the sum s, if $\lim_{n\to\infty} t_n$ exists and equals s.

The condition of regularity of the method of summability (N, p_n) defined by (1.1) are

$$\lim_{n\to\infty} p_n/P_n = 0, \tag{1.2}$$

and

$$\sum_{k=0}^{n} |p_k| = O(P_n), \quad \text{as} \quad n\to\infty. \tag{1.3}$$

If $\{p_n\}$ is real and non-negative, (1.3) is automatically satisfied and then (1.2) is the necessary and sufficient condition for the regularity of the method of summation (N, p_n).

In the special case in which $p_n = 1/(n+1)$, and, therefore

$$P_n \sim \log n, \quad \text{as} \quad n\to\infty,$$

t_n reduces to the familiar 'harmonic mean' [4] of s_n, and if it be denoted by t'_n, then $\sum a_n$ or the sequence $\{s_n\}$ is said to be summable by harmonic means, or summable (H), to the sum s if $\lim_{n\to\infty} t'_n = s$.

If the method of summability (N, p_n) be superimposed on the Cesàro means of order one, another method of summability $(N, p_n)\cdot C_1$, is obtained [1].

2. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{2.1}$$

and its conjugate series is