99. On the Sonnenschein Methods of Summability

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The class of summability methods introduced by Sonnenschein [5] is of relatively recent origin. The Sonnenschein methods, whose definition follows, include in their collection the Euler methods (E, p) from among the Hausdorff methods, the Hardy-Littlewood-Fekete circle methods (T, α) which are quasi-Hausdorff methods and also the Laurent series methods (S, α) of Meyer-König [2] and Vermes [6]. Trivially they include also the identity method, which is a member of each of the classes of Hausdorff, quasi-Hausdorff, and Nörlund methods as also of the class of (S^*, μ) methods of the present author [3] and of the Karamata methods $K[\alpha, \beta]$. (For the definition of the Karamata methods, see, for instance Sledd [4]). We show here that the Sonnenschein methods have no other methods in common with any of the five classes mentioned above. The catalyst for the enquiry is the result of Agnew [1] that the Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hausdorff methods.

Brief definitions of the various methods follow. All the methods we consider are matrix methods, provided by matrices of the type $A=(a_{nk})$, n denoting the row-index and k, the column index.

The Sonnenschein methods are defined by matrices $F=(f_{nk})$, related to a function $f(z)$, regular in $|z|<R, R\geq 1, f(1)=1$ and the elements of the matrix are given by

$$[f(z)]^n=\sum_{k=0}^{\infty} f_{nk}z^k, \text{ for each } n, \text{ with } f_{00}=1 \text{ and } f_{0k}=0, k\neq 0.$$

The Hausdorff method (H, μ) is defined by the matrix $H=(h_{nk})$ where

$$h_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k,\ (n\geq k), \text{ and } =0\ (n<k).$$

The quasi-Hausdorff method (H^*, μ) is given by the matrix $H^*=(h^*_{nk})$ where

$$h^*_{nk}=0,\ (n>k) \text{ and } =\binom{k}{n}\Delta^{k-n}\mu_n,\ (n\leq k).$$

The (S^*, μ) methods are given by the matrix $S^*=(s^*_{nk})$, with

$$s^*_{nk}=\binom{n+k}{k}\Delta^n\mu_k \text{ for all } n \text{ and } k.$$

The Euler method (E, p), the circle method (T, p), the Laurent

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