91. Boundary Convergence of Blaschke Products in the Unit-Circle

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(1) **Introduction.** Let $B(z)$ be Blaschke products:

$$B(z)=\prod_{n=1}^{\infty} b(z, a_n),$$

where

$$b(z, a)=\bar{a}/|a|\cdot(a-z)/(1-\bar{a}z),$$
$$0<|a_n|<1 \quad (n=1,2,\cdots),$$
$$\sum_{n=1}^{\infty}(1-|a_n|)<+\infty. \tag{1.1}$$

In this note, we shall establish the following two theorems on boundary convergence of Blaschke-products.

Theorem 1 is concerned with the necessary and sufficient condition for $B(z)$ to be regular at $z=e^{i\theta}$:

Theorem 1. *If $z=e^{i\theta}$ is not the limiting point of $\{a_n\}$, then $B(z)$ is absolutely and uniformly convergent to a regular function in the neighborhood of $z=e^{i\theta}$.*

As its immediate consequences, we get

Corollary 1. *For $B(z)$ to be singular at $z=e^{i\theta}$, it is necessary and sufficient that $z=e^{i\theta}$ is the limiting point of $\{a_n\}$.*

Corollary 2. *If $B(z)$ is regular at $z=e^{i\theta}$, then $B(z)$ is uniformly and absolutely convergent in the neighborhood of $z=e^{i\theta}$.*

In the preceding paper ([2] 4–5), the author proved Corollary 1 by somewhat complicated method.

Theorem 2 is of Abelian type:

Theorem 2. *If $B(z)$ is absolutely convergent at $z=e^{i\theta}$, then $B(z)$ tends uniformly to $B(e^{i\theta})$ as $z\to e^{i\theta}$ within Stolz-domain with vertex at $z=e^{i\theta}$.*

As its consequence, we have

Corollary 3. *If $B(z)$ is absolutely convergent at $z=e^{i\theta}$, then $B(re^{i\theta})$ is continuously defined for $0\leqq r\leqq+\infty$ by the unique formula: $\prod_{n=1}^{+\infty} b(re^{i\theta}, a_n)$.*

Corollaries 2 and 3 are remarkable phenomena, whose analogy in the case of Taylor series cannot exist evidently.

(2) **Proof of Theorem 1.** By the simple computation,

$$B(z)=\prod_{n=1}^{+\infty}\{1+c(z, a_n)\}, \tag{2.1}$$