131. The Kernel Representation of the Fractional Power of the Elliptic Operator

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§1. **Introduction.** Let A be a strongly elliptic operator defined in a domain D of R^n, and let us consider the Dirichlet problem for the operator $A+\lambda I$, λ be a complex number. Then we can define the fractional power $A^{-\alpha}$ under a suitable condition on the spectrum of A. In the case where A is formally self-adjoint, T. Kotake and M. S. Narasimhan [2] have recently proved that $A^{-\alpha}(\mathrm{Re}\,\alpha>0)$ has a kernel representation and moreover this kernel is very regular. In this article, we want to obtain the same result for not always self-adjoint operator. We consider the Dirichlet problem in the space $L^2(D)$. We express the weak solution $u\in L^2(D)$ of the equation $Au+\lambda u=f\in L^2(D)$ by means of parametrix according to H. G. Garnir [1], and we also express the Green kernel of $A+\lambda I$ using the Green operator G_λ. Finally, we show that the kernel $K^{(\alpha)}$ of $A^{-\alpha}$ is very regular. To show this, we used some properties of parametrix which are due to S. Mizohata [3]. The detailed proof will be given in a forthcoming paper.

I thank here Prof. Mizohata, who encouraged me in this subject.

§2. **Expression of solutions.** Let us consider the strongly elliptic partial differential operator of order $2m$ defined in a domain D (bounded or unbounded) of R^n

$$A=A\Big(x,\frac{\partial}{\partial x}\Big)=\sum_{|\nu|\leqq 2m}a_\nu(x)\Big(\frac{\partial}{\partial x}\Big)^\nu,\ \text{where} \tag{2.1}$$

$$\Big(\frac{\partial}{\partial x}\Big)^\nu=\Big(\frac{\partial}{\partial x_1}\Big)^{\nu_1}\Big(\frac{\partial}{\partial x_2}\Big)^{\nu_2}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\nu_n}.$$

The coefficients $a_\nu(x)$ belong to $\mathscr{B}(\widetilde{D})$, where $\widetilde{D}$ is an open set such that $\overline{D}\subseteq\widetilde{D}$. The condition of ellipticity

$$\mathrm{Re}\sum_{|\nu|=2m}a_\nu(x)(iy)^\nu\geqq\gamma|y|^{2m},\ \text{for all } y\in R^n,\ \gamma\text{: const.}>0, \tag{2.2}$$

is to be fulfilled uniformly in D. We denote by $A'=A'\Big(x,\dfrac{\partial}{\partial x}\Big)$ the transposed operator of A. Because we only need the local expressions (expressions in a fixed compact set contained in D) of weak solutions and of Green kernels, without loss of generality we can suppose that the coefficients $a_\nu(x)$ are defined in R^n and the uniform ellipticity (2.2) holds in R^n as well.

At first, we assume the existence of the parametrix E of A (resp.