130. On an Example of Non-uniqueness of Solutions of the Cauchy Problem for the Wave Equation

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1. **Introduction.** In the recent note [4] F. John has constructed the following example: For any positive integer m there exists a solution of the wave equation $\Box u=(\partial^2/\partial x^2+\partial^2/\partial y^2-\partial^2/\partial t^2)u=0$, which is analytic in a cyrindrical domain $\mathcal{D}=\{(x, y, t); x^2+y^2<1\}$ and belongs to C^m in R^3 not C^{m+2} in the neighborhood of any point outside $\mathcal{D}$.

The purpose of this note is to construct real valued functions u, f and g which belong to $\mathcal{B}$ and satisfy the equation $Lu\equiv(\Box+f\,\partial/\partial t+g)u=0$ in R^3, where the support of u equals to the set $R^3-\mathcal{D}$. What is remarkable is that the cylinder $S=\{(x, y, t); x^2+y^2=1\}$ is non-characteristic for L. Hence this example shows that for the operator L the uniqueness of solutions of the Cauchy problem for the non-characteristic surface S does not hold. But we must remark that any solution for the equation with the principal part $\Box$, which has its support in a 'strictly convex set' at a point of a time-like plane, vanishes identically in a neighborhood of that point (see [5]).

Many examples of non-uniqueness have been constructed by A. Pliś [6] and [7], P. Cohen [1] etc., and L. Hörmander has proved in the general theory that the uniqueness for an operator with the principal part $\Box$ does not hold even for a time-like plane if we admit complex valued coefficients (see [3] p. 228). But our example is interesting in the physical meaning and we can take $f=0$ if we admit complex valued g and u.

We shall construct this by the method of A. Pliś [7], using the asymptotic expansion of Bessel functions $J_\lambda(\lambda a)$ in the interval $(0, 1-\lambda^{-2\rho/5}]$ for a fixed ρ $(0<\rho<1)$.

2. **Lemma 1.** *Let $J_\lambda(a)$ be Bessel functions of order $\lambda>0$. Then, for any fixed $\rho(0<\rho<1)$ we have the following asymptotic formula:*

$$(1)\qquad J_\lambda(\lambda a)=(2\pi\lambda\tanh\alpha)^{-1/2}\exp\{\lambda(\tanh\alpha-\alpha)\}(1+0(\lambda^{-1/5}))$$
$$(0<a<1,\ \cosh\alpha=a^{-1},\ \alpha>0)$$

which is valid uniformly for every a in $(0, 1-\lambda^{-2\rho/5}]$.

Proof. First of all we remark

$$(2)\qquad 1\geqq\tanh\alpha=\sqrt{1-a^2}\geqq\lambda^{-\rho/5}\ \text{ in }\ 0<a\leqq1-\lambda^{-2\rho/5}.$$

We shall use a well-known integral representation of Bessel functions (see [2] p. 412):