129. A Note on the Logarithmic Means

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§ 1. When a sequence $\{s_n\}$ is given we define the logarithmic means by the transformation

$$t_0=s_0,\quad t_1=s_1,$$

$$(1)\qquad t_n=\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\quad (n\geq 2).$$

If $\{t_n\}$ tends to a finite limit s as $n\to\infty$, we shall denote that $\{s_n\}$ is summable (l) to s. (See [2] p. 59, p. 87.)

As is well known the Cesàro means $(C, 1)$ are defined by the transformation

$$(2)\qquad \sigma_n=\frac{1}{n+1}(s_0+s_1+\cdots+s_n)\quad (n\geq 0).$$

Concerning these methods of summability we know the following

Theorem 1. *If $\{s_n\}$ is summable $(C, 1)$ to s, then it is summable (l) to the same sum. There is a sequence summable (l) but not summable $(C, 1)$.* (See [2] p. 59, [7] p. 32.)

We shall prove, in this note, some converse of this theorem.

Theorem 2. *If $\{s_n\}$ is summable (l), with*

$$\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)=s+o\left(\frac{1}{\log n}\right),$$

then $\{s_n\}$ is also summable $(C, 1)$. The condition $o\left(\frac{1}{\log n}\right)$ cannot be replaced by $O\left(\frac{1}{\log n}\right)$.

Proof. From (1) and (2) we get

$$s_0=t_0,\quad s_1=t_1,\quad s_2=3\left(t_2\log 2-t_0-\frac{t_1}{2}\right),$$

$$s_n=(n+1)\{t_n\log n-t_{n-1}\log(n-1)\}\quad (n\geq 3),$$

and

$$(3)\qquad \begin{aligned}\sigma_n&=\frac{1}{n+1}(s_0+s_1+\cdots+s_n)\\ &=\frac{-1}{n+1}\left\{2t_0+\frac{1}{2}t_1+t_2\log 2+t_3\log 3+\cdots+\right.\\ &\qquad\left.+t_{n-1}\log(n-1)\right\}+t_n\log n.\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{-1}{n+1}\left\{2+\frac{1}{2}+\log 2+\log 3+\cdots+\log(n-1)\right\}+\log n$$