125. On the Product of Paracompact Spaces

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1. Introduction. As is well known, the topological product of two paracompact Hausdorff spaces is not normal in general. The cases for which the topological product $X\times Y$ of a Hausdorff space X with any paracompact Hausdorff space Y has been proved to be normal are as follows:

(a) X is compact (J. Dieudonné [1]).

(b) X is σ-compact and regular (E. Michael [3]).

(c) X is paracompact and locally compact (K. Morita [7]).

In this paper we shall show that these cases can be unified into a single case. Namely, we shall establish the following theorem

Theorem 1. *Let X be a paracompact normal space which is a countable union of locally compact closed subsets, and let Y be a paracompact normal space. Then the product space $X\times Y$ is paracompact and normal.*[1)]

As an example of a paracompact normal space which is a countable union of locally compact closed subsets we can mention a CW-complex in the sense of J. H. C. Whitehead [16]. It is known (cf. C. H. Dowker [2, p. 563]) that the topological product of two CW-complexes is a closure finite cell complex but not a CW-complex in general. Theorem 1 shows that not only the product of two CW-complexes but also the product of a CW-complex with any paracompact normal space is paracompact and normal.

In Theorem 1 the condition that X be a countable union of locally compact closed subsets cannot be weakened further, at least so long as X is an M-space. Indeed, we have the following theorem, which gives a partial answer to a problem raised by H. Tamano [15].

Theorem 2. *Let X be an M-space, or more generally, a countable union of closed subsets each of which is an M-space. Then in order that the product space $X\times Y$ be normal for any paracompact normal space Y it is necessary and sufficient that X be a paracompact normal space which is a countable union of locally compact closed subsets.*

The notion of M-spaces was introduced and discussed in our previous paper [11]. Countably compact spaces, metrizable spaces, and

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1) It should be noted that the Hausdorff or T_1 separation axiom is not assumed for paracompact normal spaces throughout this paper.