# 147. *On the Point Spectrum of the Schrödinger Operator*

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1. **Introduction.** Let us consider the Schrödinger operator defined in $R^3$

$$L=\sum_{j=1}^{3}\left(\frac{1}{i}\frac{\partial}{\partial x_j}-b_j(x)\right)^2+q(x) \tag{1.1}$$
$$\equiv -\Delta+2i\sum b_j\frac{\partial}{\partial x_j}+i\sum\frac{\partial b_j}{\partial x_j}+c(x),$$

where $b_j(x)$ and $q(x)$ are real-valued. Our purpose is to show that, under certain conditions on $b_j$ and $q$, the point spectrum of the operator $L$ is finite.

Let us assume[1)]

$$(\mathrm{C}_1)\quad b_j(x)\in\mathcal{B}^1(R^3),\quad c(x)\in\mathcal{E}^0(Co),\quad |c(x)|\leq\frac{\text{const}}{|x|^{1.5-\varepsilon}}+\text{const},\quad \varepsilon>0.$$

Under this assumption, it is easy to see

**Lemma 1.1.** *The operator $L$ has a unique self-adjoint extension $A$, and $\mathcal{D}(A)=\mathcal{D}^2_{L^2}$, moreover we have*

$$\|u(x)\|_{\mathcal{D}^2_{L^2}}\leq C(\Lambda)\|u\|_{L^2} \tag{1.2}$$

*for any eigenfunction $(\lambda-A)u=0$ for $\lambda\leq\Lambda$, $\Lambda$ being arbitrary positive number.*

In section 2, we require more stringent condition:

$$(\mathrm{C}_2)\quad b_j(x)\in\mathcal{E}^2(R^3);\ b_j(x),\ |x|\frac{\partial b_j}{\partial x_i}(x)\ \text{are bounded};\ c(x)\in\mathcal{E}^1(Co);$$
$$|x|\cdot\left|\frac{\partial c}{\partial x_i}(x)\right|\leq\frac{\text{const}}{|x|^{1.5-\varepsilon}}+\text{const},\quad \varepsilon>0.$$

Then, under the assumptions $(\mathrm{C}_1)$ and $(\mathrm{C}_2)$, we have

**Lemma 1.2.** *Let $u(x)\in\mathcal{D}^2_{L^2}$ be a solution of $Au=\lambda u$, $\lambda$ real. We have $u(x)\in\mathcal{E}^3_{L^2(\mathrm{loc})}(Co)$. Moreover, in a neighbourhood of the origin, we have*

$$|u(x)|\leq\text{const},\quad |u_{x_i}(x)|\leq\frac{\text{const}}{|x|^{0.5-\varepsilon}},\quad |u_{x_ix_j}(x)|\leq\frac{\text{const}}{|x|^2}.$$

2. **Upper boundedness of the eigenvalues.**

**Theorem 1.** *Under the assumptions* $(\mathrm{C}_1)$, $(\mathrm{C}_2)$, *there exists a* $\lambda_0>0$

1) In this note, we used the notations of L. Schwartz in his treatise (Théorie des Distributions). Let us explain these briefly: $f(x)\in\mathcal{B}^m$, if $f(x)$ has continuous bounded derivatives up to order $m$. $f(x)\in\mathcal{E}^m(\Omega)$, if $f$ is merely continuously differentiable in $\Omega$ up to order $m$. $\mathcal{D}^m_{L^2}$ is the space of all functions such that $D^\nu f\in L^2(R^n)$, $|\nu|\leq m$, $\|f\|^2_{\mathcal{D}^m_{L^2}}=\sum_{|\nu|\leq m}\|D^\nu f\|^2_{L^2}$. $\mathcal{E}^m_{L^2}(\Omega)$ is the space of all functions such that $D^\nu f(x)\in L^2(\Omega)$, $|\nu|\leq m$, with the norm: $(\sum_{|\nu|\leq m}\|D^\nu f\|^2_{L^2(\Omega)})^{\frac{1}{2}}$. $f\in\mathcal{E}^m_{L^2(\mathrm{loc})}(\Omega)$, if $\alpha f\in\mathcal{E}^m_{L^2}(\Omega)$, for all $\alpha(x)\in\mathcal{D}(\Omega)$.