144. A Note on the Functional-Representations of Normal Operators in Hilbert Spaces

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Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let both $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ be incomplete orthonormal infinite sets which are orthogonal to each other and by which a complete orthonormal system in $\mathfrak{H}$ is constructed; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed bounded sequence of complex numbers; let (u_{ij}) be an infinite unitary matrix with $|u_{jj}|<1, j=1, 2, 3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let N be the operator defined by

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu(x, \varphi_\nu)\varphi_\nu+c\sum_{\mu=1}^{\infty}(x, \psi_\mu)\Psi_\mu$$

for every $x\in\mathfrak{H}$ and an arbitrarily given complex constant c; let L_y be the continuous linear functional associated with an arbitrary element $y\in\mathfrak{H}$; and let the operator N defined above be denoted symbolically by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}.$$

Then Nx is expressible in the form

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x)\quad(x\in\mathfrak{H}).$$

In Proceedings of the Japan Academy, Vol. 37, 614–618 (1961), I defined "the functional-representation of N" by $\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}$ and proved that the functional-representation of N converges uniformly, that N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$, and that $\|N\|=\max(\sup_\nu|\lambda_\nu|, |c|)$. In the same Proceedings, Vol. 38, 18–22 (1962), conversely I treated of the question as to whether any bounded normal operator with point spectrum in $\mathfrak{H}$ can always be expressed in the form of the above-mentioned infinite series of the continuous linear functionals associated with all the elements of a complete orthonormal system in $\mathfrak{H}$, by using such a unitary matrix as above. Though, in the latter paper, the conclusion was affirmative, an additional hypothesis, that is, "If the whole subset with non-zero measure of the continuous spectrum of N lies on a circumference with center at the origin" had to be set up: for otherwise, in the particular case where N has no eigenvalue, N is not necessarily expressed by the linear combination of L_{ψ_μ} in connection with the unitary