142. The Number of Tree Semilattices

By Tôru SAITÔ
Tokyo Gakugei University

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By a *tree semilattice* we mean a semilattice T which satisfies the following condition:

if $a \leqq a'$, $b \leqq b'$, a and b are non-comparable, then a' and b' are non-comparable.

This semilattice plays an important role in the theory of ordered semigroups ([1], [2]).

For a positive integer n, we denote by $T(n)$ the number of non-isomorphic tree semilattices of order n. In this note we give a method of calculating the number $T(n)$.

Let T be a tree semilattice of order n and let 0 be the zero element of T. We denote $T \setminus 0$ by T'. Clearly $T(1)=1$. If $n>1$, then T' is decomposed into disjoint tree semilattices, say, i tree semilattices of order 1, j tree semilattices of order 2, k tree semilattices of order 3 and so on. Evidently

$$n-1=i+2j+3k+\cdots .$$

Now there is 1 way of selecting i tree semilattices of order 1, ${}_{T(2)}H_j$ ways of selecting j tree semilattices of order 2, ${}_{T(3)}H_k$ ways of selecting k tree semilattices of order 3 and so on. Thus we have the following

Theorem. *$T(n)$ satisfies the formal relation*

$$(1+x+x^2+\cdots)(1+{}_{T(2)}H_1x^2+{}_{T(2)}H_2x^4+{}_{T(2)}H_3x^6+\cdots)$$
$$(1+{}_{T(3)}H_1x^3+{}_{T(3)}H_2x^6+{}_{T(3)}H_3x^9+\cdots)$$
$$=T(1)+T(2)x+T(3)x^2+T(4)x^3+T(5)x^4+\cdots .$$

Comparing the corresponding coefficients, we can calculate $T(n)$ recursively. We list the first 10 numbers of $T(n)$.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	1	2	4	9	20	48	115	286	719

Appendix 1. Tree semilattice in the above sense were called flowing semilattice by Tamura [3].

2. In [4], Kimura gave a formula to calculate the number of orderable semilattices (in his sense). Reminding of Theorem 14 in [1], this formula can be obtained by a similar reasoning as in the present paper.