140. *Semigroups Whose Arbitrary Subsets Containing a Definite Element are Subsemigroups*

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(Comm. by Kenjiro SHODA, M.J.A., Nov. 12, 1963)

1. Consider a semigroup S satisfying the following condition: Any subset of S which contains a definite element e is a subsemigroup of S.

A semigroup S is called a β^*-semigroup if S satisfies the above condition.

For example semigroups of order 2, β-semigroups [4][1] and Rédei's semigroups are all β^*-semigroups, where by a Rédei's semigroup we mean a semigroup satisfying the condition that any non-empty subset is a subsemigroup [2].[2]

2. Immediately we have that a homomorphic image of S is a β^*-semigroup and any subset of S which contains e is also a β^*-semigroup.

Putting now $T=\{x\in S;\ x^2=x\}$, $U=\{x\in S;\ x^2=e,\ x\neq e,\ ex=xe=e\}$, and $V=\{x\in S;\ x^2=e,\ x\neq e,\ ex=xe=x\}$, it follows that V has at most one element and $S=T+U+V$ (disjoint class-sum).

We define a relation $\approx$ as follows:

$a\approx b$ means that at least one of $a\underset{l}{\sim}b$, $a\underset{r}{\sim}b$ and $a\sim b$ holds, provided that $a\underset{l}{\sim}b$ [$a\underset{r}{\sim}b$] means $ab=a$ and $ba=b$ [$ab=b$ and $ba=a$] for a, b in T, $a\sim b$ does $ab=ba=e$ for a, b in $S\setminus T$.[3]

Then we have the following lemmas.

Lemma 1. $\approx$ is an equivalence relation defined in S.

Lemma 2. For any a, b in U, any c in T and w in V $a\approx b$, $w\not\approx a$ ($\not\approx$ denotes the negation of $\approx$), $w\not\approx c$ and $a\not\approx c$.

Lemma 3. If $V\neq\square$,[4] then $e\approx a$ implies $e=a$.

Thus we have

Theorem 1. S can be represented as

$$S=\sum_{\alpha\in\Lambda}S_\alpha=\sum_{\lambda\in\Lambda_l}S_\lambda+\sum_{\mu\in\Lambda_r}S_\mu+\sum_{\nu\in\Lambda_0}S_\nu \text{ (disjoint class-sum)}$$

where $\Lambda=\Lambda_l\smile\Lambda_r\smile\Lambda_0$, $\Lambda_0=\{\omega, \varepsilon, \upsilon\}$,

S_λ, $\lambda\in\Lambda_l$ [S_μ, $\mu\in\Lambda_r$] is a maximal left [right] zero[5] subsemigroup which contains no e,

1) The numbers in brackets refer to the references at the end of the paper.

2) See Theorem 50 in [2].

3) $S\setminus T$ means the set of all elements belonging to S but not to T.

4) $\square$ denotes the empty set.

5) A left [right] zero is a semigroup defined by $xy=x$[$xy=y$] for all x, y.