139. A Note on Intra-regular Semigroups

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Following Clifford and Preston [2] we shall say that a semigroup S is intra-regular if, for any element a of S, there exist x and y in S such that

$$xa^2y=a,$$

that is, S is intra-regular if $a\in Sa^2S$, for all a in S.

It is easy to see that e.g. every semigroup admitting relative inverses (see: [1]) is an intra-regular semigroup. A subset X of a semigroup S is called semiprime if $a^2\in X$, $a\in S$ imply $a\in X$. It is known that a semigroup S is intra-regular if and only if every two-sided ideal of S is semiprime (see: [2] Lemma 4.1). Throughout this paper, by ideal we shall mean two-sided ideal. For the fundamental concepts of the algebraic theory of semigroups we refer to [2] and [4].

In this note we prove the following

Theorem. *Every ideal of an ideal of an intra-regular semigroup S is an ideal of S.*

For the proof we need the following

Lemma 1. *In an intra-regular semigroup every ideal A is idempotent, that is, $A^2=A$.*

Proof. Let S be an intra-regular semigroup, and let A be an ideal of s. We prove that $A\subseteq A^2$. Let $a\in A$, then $a^2\in A^2$. This implies that $a\in A^2$, because A^2 is an ideal of S, and every ideal of S is semiprime. Since $A^2\subseteq A$, we obtain that $A^2=A$. Thus every ideal of S is idempotent.

Lemma 2. *Let S be an arbitrary semigroup, and let B be an ideal of an ideal a of S. Denote by $\overline{B}$ the ideal of S generated by B. Then $\overline{B}^3\subseteq B$.*

Proof. Since

$$\overline{B}=B\cup BS\cup SB\cup SBS,$$

it follows that

$$\overline{B}^3\subseteq A\overline{B}A=A(B\cup BS\cup SB\cup SBS)A\subseteq B,$$

which we wished to prove.

Now we can prove the above theorem. Let S be an intra-regular semigroup. Let A be an ideal of S, and let B be an ideal of the semigroup A. Denote by $\overline{B}$ the ideal of S generated by B. Then