162. A Note on the Asymptotic Behaviour of a Power Series near its Circle of Convergence

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1. Let $\sum a_n$ be an infinite series having partial sums

$$s_n = a_0 + a_1 + \cdots + a_n,$$

and let S_n^α and s_n^α denote respectively the nth Cesàro sum and mean of order α of the sequence $\{s_n\}$. Then

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu,$$

and

$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha},$$

where

$$\sum_{n=0}^{\infty} A_n^\alpha z^n = (1-z)^{-\alpha-1} \quad \text{for} \quad |z|<1,$$

so that

$$A_n^\alpha = \binom{n+\alpha}{n}$$

and, for $\alpha \neq -1, -2, \cdots,$

$$A_n^\alpha \sim n^\alpha / \Gamma(\alpha+1),$$

as $n \to \infty$.

A series $\sum a_n$ is said to be *summable* (C, α) to the sum s, if $\lim_{n\to\infty} s_n^\alpha = s$.

2. Let a complex power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

whose radius of convergence we take, for the sake of simplicity, to be unity, represent the function $f(z)$ within the circle $|z|<1$.

The function represented by the power series is always regular inside the circle. But no information, in general, can be deduced about the regularity of the function at a point of the circumference from the mere knowledge of the convergence or divergence of the series at the point.

However, in certain special cases, some results are known which relate the coefficients a_n with the analytic properties of the function $f(z)$ on the circle of convergence. Again, special methods such as Borel's method of summability (B), have been devised to associate a 'sum-function' of the power series outside the circle, and thus obtain an analytic continuation of the function.

The first theorem in this connection is the well known Abel's theo-