159. On Global Solutions for Mixed Problem of a Semi-linear Differential Equation

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1. Introduction. Let us consider the equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^3 u}{\partial t\partial x^2}=f(u)\frac{\partial u}{\partial t}+g(u) \tag{1.1}$$

in the half space $\Omega=\{(x,t);\ x,t>0\}$.

Such an equation was considered by J. Nagumo as a model of the neuron.[1] Let us limit the behaviour of the function f and g in (1.1) as follows:

$$\begin{cases} f,\ g\in C^1,\ g(0)=0,\ -K_0(u^2+1)\le f(u)\le K_1, \\ |g(u)|\le K_2(u^2+|u|) \text{ and moreover} \\ G(u)\equiv\int_0^u g(z)dz\le K_3u^2 \end{cases} \tag{1.2}$$

where K_0, K_1, K_2, K_3 are positive constants.

Now the initial and boundary data are given as follows with the compatibility conditions

$$\begin{cases} u(x,0)=u_0(x)\in\mathcal{B}^2_+\cap\mathcal{D}^1_{L^2+} & \text{for } x\ge 0, \\ u_t(x,0)=u_1(x)\in\mathcal{B}^2_+\cap\mathcal{D}^1_{L^2+} & \text{for } x\ge 0, \\ u(0,t)=\psi(t)\in C^2 \quad \text{for } t\ge 0, \end{cases} \tag{1.3}$$

$$\begin{cases} u_0(0)=\psi(0),\ u_1(0)=\psi'(0) \\ \psi''(0)-u_1''(0)=f(\psi(0))\psi'(0)+g(\psi(0)). \end{cases} \tag{1.4}$$

Then we can prove the following:

THEOREM 1. *There exists a unique solution $u(x,t)$ in Ω and* $u(x,t)$, $u_t(x,t)\in(\mathcal{B}^2_+\cap\mathcal{D}^1_{L^2+})\,[0,T]$. (Throughout this paper, we use the following notation. Let E be a topological vector space. $f(x,t)$, or simply $f(t)$ belongs to $E[0,T]$, if $f(x,t)$ is a continuous function in $t\in[0,T]$ with values in E. $\mathcal{B}^k_+$ is the topological vector space of uniformly continuous and bounded functions in $(0,\infty)$ together with their derivatives of order up to k. If we consider square integrable functions instead of uniformly continuous and bounded functions, we have $\mathcal{D}^k_{L^2+}$.)

To prove this theorem, we should obtain a priori estimates of solution and local existence theorem adapted to the step by step continuation.

2. Local existence theorem. *Let us consider the problem in $0\le t\le T$, then there exists a function $\varphi(\xi_1,\xi_2,\xi_3)$, positive and non-increasing in each argument, such that:*