# 18. A Note on Strongly Regular Rings

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Let $R$ be a ring. We recall Drazin's definitions [2]: an element $x$ of $R$ is called semi-$\pi$-regular in $R$ if a positive integer $s=s(x)$ and an element $g=g(x)$ of $R$ exist satisfying either $x^s=xgx^s$ or $x^s=x^sgx$, $R$ being itself called semi-$\pi$-regular if every element of $R$ is semi-$\pi$-regular in $R$. An element $x$ of $R$ is called strongly regular in $R$ if an element $a=a(x)$ of $R$ exists satisfying $x=x^2a$. The ring $R$ is itself called strongly regular if every element of $R$ is strongly regular in $R$. It should be noted that in a strongly regular ring $R$, $x=x^2a$ if and only if $x=ax^2$ (see [2]).

A subring $M$ of the ring $R$, following Steinfeld [7], is said to be a quasi-ideal if $RM\cap MR\subseteq M$.

The following result has been proved (see [2, 3, 5]):

Proposition. For an arbitrary ring $R$ the following conditions are equivalent:

(1) $R$ is strongly regular;

(2) $R$ is semi-$\pi$-regular and isomorphic to a subdirect sum of division rings;

(3) $R$ is semi-$\pi$-regular and contains no non-zero nilpotent elements;

(4) every quasi-ideal $M$ of $R$ is idempotent, i.e. $M^2=M$;

(5) for every right ideal $J$ and every left ideal $L$ of $R$, $JL=J\cap L\subseteq LJ$ holds.

The concept of group membership in rings was introduced by Ranum [6]. An element $a$ of the ring $R$ is said to be a group member in $R$ if $a$ is contained in a subgroup of $R$ with respect to multiplication. Evidently, the zero element of $R$ is a group member in $R$. The purpose of this note is to give a characterization of strongly regular rings $R$ in terms of group membership in $R$.

Namely, we prove the following theorem:

Theorem. For an arbitrary ring $R$, each of the five conditions (1)–(5) in the previous proposition is equivalent to the statement:

(6) each element of $R$ is a group member of $R$.

Proof. By virtue of the previous proposition, we need only to show the equivalence of the conditions (1) and (6).

(6) implies (1): Let $x$ be an element in $R$. Then $x$ is contained in a multiplicative subgroup $G$ of $R$ and hence the equation $x=x^2a$