# 16. On Existence of Linear Functionals on Abelian Groups

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1964)

In their paper [3, p. 147], S. Mazur and W. Orlicz have proved a fundamental existence theorem on linear functional in a linear space. In this Note, we shall prove now a similar theorem on Abelian groups.

Theorem. *Let $p(x)$ be a real valued subadditive functional on an Abelian group $G$, and $x(t)$ a function from an abstract set $A$ to $G$. Let $\xi(t)$ be a real valued function on $A$. Then there is a linear functional $f(x)$ satisfying*

1) $f(x)\leq p(x)$ *for all* $x\in G$,

2) $\xi(x)\leq f(x(t))$ *for all* $t\in A$

*if and only if*

$$\sum_{i=1}^{n} m_i\xi(t_i)\leq p\Big(\sum_{i=1}^{n} m_i x(t_i)\Big) \tag{1}$$

*for any finite set $t_i\in A$ and non-negative integers $m_i$, where $i=1,2,\cdots,n$ and $n=1,2,\cdots$.*

The "only if" part is evident. To prove the "if" part, we use the technique by V. Ptak. In the middle of the proof, we need the following Aumann theorem [1].

Aumann theorem. *Let $G$ be an Abelian group with a real valued subadditive functional $p(x)$,* i. e. *$p(x+y)\leq p(x)+p(y)$ and $p(0)=0$. Let $H$ be a subgroup on $G$ and $f(x)$ a linear functional on $H$,* i. e. *$f(x+y)=f(x)+f(y)$ for $x,y\in H$. If $f(x)\leq p(x)$ for all $x\in H$, then there is a linear extension $F$ of $f$ such that $F(x)\leq p(x)$ for each $x\in G$.*

An elegant proof by G. Mokobdzki is given in a note by P. Krée in the Séminaire Choquet [2]. The present writer can not approach to the original paper [1].

*Proof of Theorem.* Consider an auxiliary subadditive functional defined by

$$\tilde{p}(x)=\inf_{\substack{t_1,\cdots,t_n\\ m_1,\cdots,m_n}}\Big[p\Big(x+\sum_{i=1}^{n} m_i x(t_i)\Big)-\sum_{i=1}^{n} m_i\xi(t_i)\Big]$$

where $m_i$ $(i=1,2,\cdots,n)$ are non-negative integers. By the condition (1), we have

$$\sum_{i=1}^{n} m_i\xi(t_i)\leq p\Big(\sum_{i=1}^{n} m_i x(t_i)\Big)\leq p\Big(x+\sum_{i=1}^{n} m_i x(t_i)\Big)+p(-x).$$

Hence $p(x)$ is well defined. On the other hand, we have $-p(-x)\leq\tilde{p}(x)\leq p(x)$, so $p(0)=0$.