15. *On Absolute Summability Factors of Infinite Series*

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1. *Definitions and Notations.* Let s_n denote the n-th partial sum of a given infinite series $\sum a_n$. We write

$$t_n=\frac{1}{L_n}\sum_{\nu=1}^{n}\frac{1}{\nu}s_\nu,$$

where

$$L_n=\sum_{\nu=1}^{n}\frac{1}{\nu}\backsim\log n,\quad \text{as } n\to\infty.$$

We say that the series $\sum a_n$ is absolutely summable $\left(R,\frac{1}{n}\right)$, or summable $\left|R,\frac{1}{n}\right|$, if the sequence $\{t_n\}$ is of bounded variation,[1] that is, the series $\sum|t_n-t_{n+1}|$ is convergent. It may be observed that this method of summability is equivalent to the absolute summability method defined by means of the auxiliary sequence

$$\frac{1}{\log n}\sum_{\nu=1}^{n}\frac{1}{\nu}s_\nu\ {}^{2)}$$

known as the Riesz logarithmic mean of $\{s_n\}$.[3]

A sequence $\{\lambda_n\}$ is said to be convex[4] if

$$\Delta^2\lambda_n=\Delta^2(\lambda_n)\geq 0,\quad n=1,2,\cdots,$$

where

$$\Delta^2(\lambda_n)=\Delta(\Delta\lambda_n)=\Delta\lambda_n-\Delta\lambda_{n+1}$$

and

$$\Delta\lambda_n=\Delta(\lambda_n)=\lambda_n-\lambda_{n+1}.$$

Let $\{\lambda_n\}$ be a monotonic increasing sequence such that

$$\lambda_n\to\infty,\qquad \text{as } n\to\infty.$$

We write

$$A_\lambda(\omega)=A_\lambda^0(\omega)=\sum_{\lambda_n\leq\omega}a_n,$$

and, for $r>0$,

$$A_\lambda^r(\omega)=\sum_{\lambda_n\leq\omega}(\omega-\lambda_n)^r a_n=r\int_0^\omega(\omega-\tau)^{r-1}A_\lambda(\tau)d\tau.$$

For $r\geq 0$, we write

$$R_\lambda^r(\omega)=A_\lambda^r(\omega)/\omega^r.$$

$\sum a_n$ is said to be absolutely summable (R,λ_n,r), or summable

1) Symbolically $\{t_n\}\in BV$.

2) This can be easily seen by virtue of Lemma 3 of Iyer's paper [4], which states that the sequence $\{\omega_n\}\equiv\left\{\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\Big/\log n\right\}$ is of bounded variation, when we note that ω_n is strictly positive for $n\geq 2$.

3) Hardy [3], §4.16.

4) Zygmund [8], p. 58.