40. On Bückner's Inclusion Theorems for Hermitean Operators

By Masahiro NAKAMURA and Midori YOSHIDA
Department of Mathematics, Osaka Gakugei Daigaku
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1. Let

$$A=\int_{-\infty}^{+\infty} t\, d\, E_t \tag{1}$$

be an Hermitean operator on a Hilbert space H. For a vector u with the norm unity, the so-called *Schwarz constants* is defined by

$$a_n=(A^n u, u), \qquad n=0, 1, 2, \cdots. \tag{2}$$

It is obvious that a_n is the n-th moment of the distribution function

$$m(t)=(E_t u, u). \tag{3}$$

According to the spectral theorem, the measure dm defined by $m(t)$ is condensed on the spectrum of A.

In numerical analysis, it is sometimes important to know that a spectre of A is contained in an interval whose end points are determined by functions of the Schwarz constants. Some theorems of such a kind which are called the *inclusion theorems* are systematically obtained by Bückner, Wielandt and the others (cf. [2; § 12.5] where detailed references are included) for a completely continuous Hermitean operators. In the present note, these inclusion theorems will be generalized for an Hermitean operator with a few modifications. It may be interested that the inclusion theorems contains the well-known Krylov-Weinstein's theorem (cf. [1] and [6]) as a special case.

2. The following theorem is fundamental for Bückner's inclusion theorems:

THEOREM 1. *Each of the sets* $\{t; t\leqq a_1\}$ *and* $\{t; t\geqq a_1\}$ *contains at least one spectre of* A. *If moreover* u *is not a proper vector belonging to* a_1, *then each of the sets* $\{t; t<a_1\}$ *and* $\{t; t>a_1\}$ *contains at least a spectre of* A.

The proof of the theorem requires a minor modification of that of Bückner [2; Thm. 12.1]. If u is a proper vector belonging to a_1, then the theorem is obvious. Now, suppose that u is not a proper vector belonging to a_1. Then the measure dm cannot concentrate at a_1. Consequently, if the spectrum of A is contained in $\{t; t\leqq a_1\}$, then

$$a_1=(Au, u)=\int_{-\infty}^{\infty} t\, d(E_t u, u)=\int_{-\infty}^{a_1} t\, dm<a_1.$$