39. On Metrizability of M-Spaces

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§1. **Introduction.** Let X be a topological space. An open covering $\mathfrak{U}$ of X is said to be a *star-refinement* of another open covering $\mathfrak{V}$ of X if the covering $\{St(U, \mathfrak{U}) | U \in \mathfrak{U}\}$ is a refinement of $\mathfrak{V}$ where $St(A, \mathfrak{U})$ means the union of the sets U of $\mathfrak{U}$ such that $A \cap U \neq \phi$. A sequence $\{\mathfrak{U}_n | n=1, 2, \cdots\}$ of open coverings of X is said to be *normal* if $\mathfrak{U}_{n+1}$ is a star-refinement of $\mathfrak{U}_n$ for $n=1, 2, \cdots$.

We shall say that a topological space X is an *M-space* if there exists a normal sequence $\{\mathfrak{U}_n | n=1, 2, \cdots\}$ of open coverings of X satisfying the condition (*) below:

(*) If a family $\mathfrak{A}$ consisting of a countable number of subsets of X has the finite intersection property and contains as a member a subset of $St(x_0, \mathfrak{U}_n)$ for every n and for some fixed point x_0 of X, then $\bigcap\{\bar{A} | A \in \mathfrak{A}\} \neq \phi$.

Metrizable spaces and countably compact spaces are clearly M-spaces.

The notion of M-spaces was introduced and discussed in [5].

Theorem 1. *Let X be a topological space. In order that X be metrizable it is necessary and sufficient that X be a paracompact Hausdorff M-space and that the product space $X \times X$ be perfectly normal.*

More precisely, we shall obtain the theorem below:

Theorem 1′. *Let X be a topological space. In order that X be metrizable it is necessary and sufficient that X be a paracompact Hausdorff M-space and that the diagonal Δ of the product space $X \times X$ be a G_δ-set in $X \times X$.*

It is easily seen that Theorem 1 is deduced from Theorem 1′. Therefore, we have only to prove Theorem 1′; this will be done in §2.

A completely regular space X is said to be *absolute* G_δ if it is a G_δ-set in every extension of it, that is, if X is a dense subset of a completely regular space Y, then X is a G_δ-set in Y.

It is well known that a metrizable space is absolute G_δ if and only if it is completely metrizable (cf. [1]).

Z. Frolik has proved that a paracompact normal space which is absolute G_δ is an M-space. More generally, K. Morita ([7], [8]) has proved that a paracompact normal space which is G_δ in a countably compact space is an M-space.