36. A Property of Green's Star Domain

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Let R be a hyperbolic Riemann surface and $g(p, o)$ be the Green function on R with its pole o in R. The Geen's star domain $R^{g,o}$ with respect to R and o is the set of points in R which can be joined by Green arcs issuing from o. We also assume that o is a member of $R^{g,o}$. We shall see that $R^{g,o}$ is a simply connected domain. Hence we can map $R^{g,o}$ onto the open unit circular disc by a one-to-one conformal mapping φ. We shall show that the image of a singular Green line (i.e. a Green line on which $g(p, o)$ has a positive infimum) issuing from o by φ is a Jordan curve starting from $\varphi(o)$ and terminating at a point of the unit circumference. We denote by N_φ the totality of end points on the unit circumference of image curves of singular Green lines issuing from o by the mapping φ. The main purpose of this paper is to show that N_φ is of logarithmic capacity zero.

1. Let R be a hyperbolic Riemann surface. This means that there exists the Green function $g(p, o)$ with the arbitrary given pole o in R. We define the pair $(r(p), \theta(p))$ of local functions on R by the relations

$$\begin{cases} dr(p)/r(p) = -dg(p, o) \\ d\theta(p) \qquad = -{}^*dg(p, o). \end{cases}$$

By giving the initial condition $r(o)=0$, $r(p)$ is the global function $e^{-g(p,o)}$ on R. Each branch of $r(p)e^{i\theta(p)}$ can be taken as a local parameter at each point of R except possibly a countable number of points at which $d\theta(p)=0$. A Green arc is an open arc on which $\theta(p)$ is a constant, being considered locally, and $d\theta(p)\neq 0$. A *Green line* is a maximal Green arc. We denote by $G(R, o)$ the totality of Green lines issuing from o. We set, for each $L\in G(R, o)$,

$$d(L)=\sup(r(p);\ p\in L).$$

Clearly $0<d(L)\leq 1$. We say that $L(\in G(R, o))$ is a *singular Green line* if $d(L)<1$. We denote by $N(R, o)$ the set of all singular Green lines in $G(R, o)$. We also denote by $E(R, o)$ the totality of L in $G(R, o)$ such that the closure of L contains a point $p(\neq o)$ with $d\theta(p)=0$. Clearly $G(R, o)\supset N(R, o)\supset E(R, o)$. We set

$$R^{g,o}=(o)\cup(p\in R;\ p\in L \text{ for some } L \text{ in } G(R, o)).$$

We call the set $R^{g,o}$ the *Green's star domain* with respect to R and o. Then we see that