32. *On Differentially Integral Elements*

By Hajime NISHIMURA

Institute of Mathematics, Yoshida College, Kyoto University

(Comm. by Zyoiti SUETUNA, M.J.A., March 12, 1964)

1. Let R be a differential ring with a derivation δ, and P be a differential subring of R. J. Brzeziński [1] has introduced the following notions. An element x of R is called *differentially integral* with respect to P if there exists a finitely generated P-submodule of R containing all the derivatives: $\delta^0 x=x$, δx, $\delta^2 x=\delta\cdot\delta x, \cdots$. When R is moreover an integral domain, R is said to be *differentially-integrally closed* if every element of the quotient field $Q(R)$ of R which is differentially integral with respect to R is contained in R.

If one wishes to go into the differential algebra of non-zero characteristic, it will be found that these notions may not sufficiently answer the purpose.[1] In a recent paper [3], K. Okugawa, taking up a differential ring with Hasse's *higher differentiations*, has developed the Picard-Vessiot theory for linear homogeneous differential equations, which is a generalization of the work of E. Kolchin [2] to the case of an arbitrary characteristic.

In this note, we consider the corresponding notion of the differentially integral element concerning the differential ring in the latter sense, and we generalize the condition which is given in [1], for a unique factorization domain to be differentially-integrally closed. This we discuss in a manner similar to that of [1].

2. The notation and terminology will be as in [3]. Let R be a ring.[2] A sequence $\boldsymbol{\delta}=\{\delta_\nu;\ \nu=0, 1, 2, \cdots\}$ of maps $\delta_\nu\colon R\rightarrow R$ is called a *differentiation* in R, if it satisfies

(D1) $\delta_0 x=x$, (D2) $\delta_\nu(x+y)=\delta_\nu x+\delta_\nu y$, for all ν,

(D3) $\delta_\nu(x\cdot y)=\sum_{\lambda+\mu=\nu}\delta_\lambda x\cdot\delta_\mu y$, for all ν, (D4) $\delta_\lambda(\delta_\mu x)=\binom{\lambda+\mu}{\lambda}\delta_{\lambda+\mu}x$, for all λ, μ.

A ring $\boldsymbol{R}$ with the mutually commutative differentiations $\boldsymbol{\delta}_i=\{\delta_{i,\nu};\ \nu\geqq 0\}$ $(1\leqq i\leqq m)$ preassigned in $\boldsymbol{R}$ is called a *differential ring*.[3] The set $\Theta=\{\delta_{1,\nu_1}\cdots\delta_{m,\nu_m};\ \nu_1\geqq 0, \cdots, \nu_m\geqq 0\}$ is regarded as the domain of differential operators in $\boldsymbol{R}$.

---

1) In the case of the characteristic $p>0$, since $(1/x)^p$ is differentially integral with respect to R for any $x\in R$, we see at once that R is never differentially-integrally closed except for the trivial case where R itself is a field.

2) When we speak of a *ring* in this note, we always suppose tacitly that it is a commutative ring with unity.

3) In case of a characteristic zero, if $\boldsymbol{R}$ is an integral domain containing a field then it is regarded as the usual differential ring, since $\delta_{i,\nu}=\frac{1}{\nu!}\delta_{i,1}$ and $\delta_{i,1}$ is the usual derivation in $\boldsymbol{R}$ for $1\leqq i\leqq m$. See [3], §1.