59. A Note on the Convergence of Semi-groups of Operators

By Minoru HASEGAWA

Department of Mathematics, Tokyo Metropolitan University

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1. In the following we shall deal with a sequence of one-parameter semi-groups $\{U_t^{(n)}\}$ $(t\geqq 0, n=1, 2, \cdots)$ of operators on a fixed Banach space $\mathfrak{B}$ to $\mathfrak{B}$ which satisfies the stability condition, that is,

$$U_t^{(n)}U_{t'}^{(n)}=U_{t+t'}^{(n)}\quad (t, t'\geqq 0),\ U_0^{(n)}=I,$$
$$\lim_{t\to t_0} U_t^{(n)}f=U_{t_0}^{(n)}f\quad (t_0\geqq 0, f\in\mathfrak{B}),$$
$$\|U_t^{(n)}\|\leqq Me^{\alpha t},$$

where M and α are independent of n and t.

For simplicity we assume $M=1$.

Let $\mathfrak{G}^{(n)}$ be the infinitesimal generator of $\{U_t^{(n)}\}$, that is,

$$\mathfrak{G}^{(n)}\varphi=\lim_{h\downarrow 0} h^{-1}(U_h^{(n)}-I)\varphi,$$

then the domain $\mathfrak{D}(\mathfrak{G}^{(n)})$ of $\mathfrak{G}^{(n)}$ is dense in $\mathfrak{B}$, and for any $m>\alpha$ the inverse operator $I_m^{(n)}=(I-m^{-1}\mathfrak{G}^{(n)})^{-1}$ is linear and satisfies following relations

$$I_m^{(n)}f=m\int_0^\infty e^{-mt}U_t^{(n)}f\,dt\quad (f\in\mathfrak{B}),$$
$$\|I_m^{(n)}\|\leqq(1-m^{-1}\alpha)^{-1}.$$

Our aim is to solve the problem of the following type.

Assumption (A). *$\{\mathfrak{G}^{(n)}\varphi\}_n$ is a Cauchy sequence in $\mathfrak{B}$ for any $\varphi\in\mathfrak{M}\subseteqq\bigcup_k\bigcap_{n\geqq k}\mathfrak{D}(\mathfrak{G}^{(n)})$, where $\mathfrak{M}$ is dense in $\mathfrak{B}$.*

Under Assumption (A), is it true that the additive operator $\mathfrak{G}=\lim_{n\to\infty}\mathfrak{G}^{(n)}$ or some closed extension of $\mathfrak{G}$ is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t=\lim_{n\to\infty}U_t^{(n)}$?

Our main theorem Theorem 2 is an answer to this problem.

The following theorem had been treated by H. F. Trotter [1].

Theorem 1. *Under Assumption* (A), *the closure $\widetilde{\mathfrak{G}}$ of $\mathfrak{G}$ is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t=\lim_{n\to\infty}U_t^{(n)}$ if and only if the following Condition* (A_1) *is satisfied.*

Condition (A_1). *For some $m>\alpha$, the range $\mathfrak{R}(I-m^{-1}\mathfrak{G})$ of $I-m^{-1}\mathfrak{G}$ is dense in $\mathfrak{B}$.*

As an application we shall treat this theorem from above general point of view and prove Theorem 1 by using Theorem 2.

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