54. A Note on the Galois Cohomology Group of the Ring of Integers in an Algebraic Number Field

By Hideo YOKOI

Nagoya Institute of Technology, Nagoya

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1. **Introduction.** Let K be a finite Galois extension of a finite algebraic number field F and let $G=G(K/F)$ be the Galois group of K/F. Denote by $\mathfrak{o}_K$ and $\mathfrak{o}_F$ the rings of integers in K and F respectively. As usual, we shall denote by $H^r(G, A)$ the r-dimensional Galois cohomology group of G acting on a G-module A. Following Artin-Tate-Chevalley, we shall consider $H^r(G, A)$ also for negative r.

In (1) we proved the following

Theorem 1. *If we assume that the 0-dimensional Galois cohomology group $H^0(G, \mathfrak{o}_K)$ of $\mathfrak{o}_K$ with respect to K/F is trivial, then the Galois cohomology group of $\mathfrak{o}_K$ with respect to K/Ω is trivial for every dimension and for any intermediate field Ω of K/F.*

Later we obtained in (2) and (3) the following

Theorem 2. *Let K/F be a cyclic extension of prime order p. Then, for every dimension r, all the Galois cohomology groups $H^r(G, \mathfrak{o}_K)$ of $\mathfrak{o}_K$ with respect to K/F are isomorphic with each other.*

From these results, it is generally conjectured that all the Galois cohomology groups $H^r(G, \mathfrak{o}_K)$ of $\mathfrak{o}_K$ with respect to K/F have the same order. In this note we shall prove that this is in fact the case if K/F is a cyclic extension of any finite degree.

2. Let F be an algebraic number field of degree m and let K/F be a cyclic extension of degree n. Denote by $G=G(K/F)$ the Galois group of K/F. Then there exists a number B in K, by the theorem on existence of normal basis,[1] such that the conjugates $B^{(0)}, B^{(1)},\cdots, B^{(n-1)}$ of B form a basis of K over F, i.e. a normal basis of K/F. Since we may choose an integer c such that cB becomes an integer in K, we can assume from the beginning, without losing generality, that B is an integer in K.

Further, let $\{\omega_1, \omega_2,\cdots, \omega_m\}$ be an arbitrary integral basis of F, and denote by $\mathfrak{o}^*$ the module generated by $\omega_i B^{(j)}$ $(i=1, 2,\cdots, m;\ j=0, 1,\cdots, n-1)$. Since $\omega_i B^{(j)}$ $(i=1, 2,\cdots, m;\ j=0, 1,\cdots, n-1)$ are linearly independent over the rational number field Q, the rank of the module $\mathfrak{o}^*$ is $N=mn$, and $\mathfrak{o}^*=\mathfrak{o}_F B^{(0)}+\mathfrak{o}_F B^{(1)}+\cdots+\mathfrak{o}_F B^{(n-1)}$ is a direct decomposition of the module $\mathfrak{o}^*$. Here, $\mathfrak{o}_F$ means the module of all integers

1) Cf. e.g. E. Noether [4], M. Deuring [5] etc.