107. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XII*

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In the preceding papers we have been concerned with the general method of constructing normal operators with arbitrarily prescribed point spectra in the complex abstract Hilbert space $\mathfrak{H}$ being complete, separable, and infinite dimensional and with its applications to the theory of functions of a complex variable. In the present paper, however, we shall first set ourselves the problem of constructing normal operators with arbitrarily prescribed continuous spectra in $\mathfrak{H}$.

Theorem 29. Let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrary bounded infinite sequence of complex numbers (counted according to the respective multiplicities), and D an arbitrary connected close set with positive finite measure in the complex plane such that any point of the closure of $\{\lambda_\nu\}$ is not contained in it. Then there are infinitely many bounded normal operators N in $\mathfrak{H}$ such that the point spectrum and the continuous spectrum of each N are given respectively by $\{\lambda_\nu\}$ and the union of D and the set of all those accumulation points of $\{\lambda_\nu\}$ which do not belong to $\{\lambda_\nu\}$ itself.

Proof. Let $\varDelta$ be a Lebesgue-measurable set of positive finite measure $m(\varDelta)$ in the complex λ-plane such that it contains D as its proper subset with measure less than $m(\varDelta)$, and $L_2(\varDelta)$ the Lebesgue functionspace associated with $\varDelta$. If we consider the operator T defined by $(Tf)(\lambda)=\lambda f(\lambda)$ for every $f \in L_2(\varDelta)$, then it can be verified without difficulty that the adjoint operator T^* of T is given by $(T^*f)(\lambda)=\bar{\lambda}f(\lambda)$ and that T is a bounded normal operator in $L_2(\varDelta)$ such that its continuous spectrum and its point spectrum are given by $\bar{\varDelta}$ and the empty set respectively. Suppose now that $\{K(\lambda)\}$ denotes the complex spectral family of T and that $\{\hat{\varphi}_\nu(\lambda)\}$ and $\{\hat{\psi}_\mu(\lambda)\}$ are arbitrarily chosen orthonormal sets determining the subspaces $K(\bar{\varDelta}-D)L_2(\varDelta)=\widehat{\mathfrak{M}}$ and $K(D)L_2(\varDelta)=\widehat{\mathfrak{N}}$ respectively. If we divide D in n disjoint subsets $D_1, D_2, \cdots, D_n$ with positive measure, then $K(D_i)K(D_j)=0$ $(i \neq j;\ i,j=1,2,3,\cdots,n)$ and any $K(D_j)$ is never the null operator because of the fact that D_j belongs to the continuous spectrum of T. Hence the dimension of the space $K(D)L_2(\varDelta)$ is greater than n, however large n may be. The same is true of $K(\bar{\varDelta}-D)L_2(\varDelta)$. Since, moreover, $L_2(\varDelta)=\widehat{\mathfrak{M}}\oplus\widehat{\mathfrak{N}}$, it turns out from these results that $\{\hat{\varphi}_\nu(\lambda)\}$ and