142. On a Construction of Annihilating Spaces

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1. Throughout this note we will use the notations and results in a previous paper: *Annihilators of von Neumann Algebras* (*Annihilating Spaces*), Bull. Kyushu Inst. Tech., (M. & N.S.), No. 10, pp. 25–39 (1963). We will quote it, whenever necessary, as [A. S.].

The trace-class (τc) of operators on a Hilbert space $\mathfrak{H}$ is a Banach space with the norm $\tau(A)$ for every $A \in (\tau c)$. We shall denote by $t(A)$ the trace on (τc) and by $(\tau c)_0$ a closed subspace $\{A \mid t(A)=0\}$ of (τc). And every operator of rank ≤ 1 on $\mathfrak{H}$ is represented by $f \otimes \bar{g}$ for $f, g \in \mathfrak{H}$. Hence we have $t(f \otimes \bar{g}) = \langle f, g \rangle$.

Let $\mathfrak{T}$ be a closed subspace of $(\tau c)_0$ generated by operators of rank ≤ 1. If we put ${}^{\mathfrak{T}}\mathfrak{M}^f = \{g \mid f \otimes \bar{g} \in \mathfrak{T}\}$, then we can easily show that ${}^{\mathfrak{T}}\mathfrak{M}^f$ is a closed linear subspace of $\mathfrak{H}$ (cf. [A. S.], p. 30). Moreover, we put ${}^{\mathfrak{T}}\mathfrak{M}_f = \mathfrak{H} \ominus {}^{\mathfrak{T}}\mathfrak{M}^f$.

DEFINITION. *A closed subspace $\mathfrak{T}$ of $(\tau c)_0$ is called an annihilating space in a Hilbert space $\mathfrak{H}$, if it satisfies the following conditions:*

(1) *$\mathfrak{T}$ is generated by operators of rank ≤ 1;*

(2) *$\mathfrak{T}$ is self-adjoint, i.e., if $A \in \mathfrak{T}$, then $A^* \in \mathfrak{T}$;*

(3) *if $g \in {}^{\mathfrak{T}}\mathfrak{M}_f$, then ${}^{\mathfrak{T}}\mathfrak{M}_g \subset {}^{\mathfrak{T}}\mathfrak{M}_f$.*

In [A. S.], we characterized the annihilator $\mathfrak{R}^{\perp}$ of a von Neumann algebra $\mathfrak{R}$ as an annihilating space (cf. [A. S., Theorem 1]). Our purpose of this note is to construct an annihilating space concretely in a sense.

2. We shall state

LEMMA. *Let $\mathfrak{R}$ be a von Neumann algebra and let $\mathfrak{R}'$ be the commutant of $\mathfrak{R}$. Then a closed subspace $\mathfrak{T}$ of $(\tau c)_0$ generated by the set $\{f \otimes \bar{g}, g \otimes \bar{f} \mid f \in E(\mathfrak{H}), g \in (I-E)(\mathfrak{H}), E \in \mathfrak{R}'\}$ is an annihilating space.*

Proof. It is clear that $\mathfrak{T}$ satisfies the conditions (1), (2) of the above Definition.

Let $\mathfrak{M}_f^{\mathfrak{R}}$ be a closed linear subspace of $\mathfrak{H}$ generated by all the Xf $(X \in \mathfrak{R})$. Hence the projection $E_f^{\mathfrak{R}}$ on $\mathfrak{M}_f^{\mathfrak{R}}$ is an element of $\mathfrak{R}'$. Therefore, by definition of $\mathfrak{T}$, $\mathfrak{H} \ominus \mathfrak{M}_f^{\mathfrak{R}} \subset {}^{\mathfrak{T}}\mathfrak{M}^f$. Consequently, we have $\mathfrak{M}_f^{\mathfrak{R}} \supset {}^{\mathfrak{T}}\mathfrak{M}_f$ for every $f \in \mathfrak{H}$.

Now we shall show an inverse inclusion. If $f \in E(\mathfrak{H})$ and $g \in (I-E)(\mathfrak{H})$ for any $E \in \mathfrak{R}'$, then we have $Tf = TEf = ETf \in E(\mathfrak{H})$ for every $T \in \mathfrak{R}$. Therefore $t(T(f \otimes \bar{g})) = \langle Tf, g \rangle = 0$ for every $T \in \mathfrak{R}$.