137. A Duality Theorem for the Real Unimodular Group of Second Order

By Nobuhiko TATSUUMA

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1964)

Let G be the real special linear group of second order. G consists of all real matrices g such that

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1.$$

The purpose of the present paper is to characterize G as a "dual" of the space of its irreducible unitary representations. This space is furnished with a law according to which the Kronecker product of any two representations is decomposed into irreducible components.

This duality may be regarded as an analogue of the Tannaka's duality theorem in the case of compact groups.

Let K be a compact group, Tannaka's duality theorem states the following. Consider the totality X of irreducible unitary representations of K. The Kronecker product $\rho\otimes\sigma$ of two elements in X is decomposed into the direct sum $\sum_j\oplus\tau_j$ of finite irreducible representations. In this decomposition, let $u\otimes v$ is equal to $\sum_j\oplus w_j$, in which u, v, w_j, are vectors in the spaces of representations ρ, σ, τ_j respectively. An element k of K decides an operator field over X, which consists of unitary matrices $\rho(k)$ in each space of representation ρ. And the decomposition of the Kronecker product $\rho(k)u\otimes\sigma(k)v$ is equal to $\sum_j\oplus\tau_j(k)w_j$. Conversely let $\{T(\rho)\}$ be an operator field over X, such that each $T(\rho)$ is a unitary matrix in the space of representation ρ, and $(T(\rho)u)\otimes(T(\sigma)v)$ is equal to $\sum_j\oplus(T(\tau_j)w_j)$, for any u and v. The duality theorem affirms that the totality of operator fields as above coincides with the original group K, that is, K is characterized as a "dual" of the space of its irreducible unitary representations, and the initial topology of K corresponds to the weakest topology which makes all the matrix element $\langle T(\rho)u, v\rangle$ continuous.

Our main theorem characterizes G in the same way as for the case of compact groups. Let Ω be the set of all equivalence classes of irreducible unitary (therefore infinite-dimensional) representations of G. We choose and fix a representation $\omega=\{U_g(\omega), \mathfrak{H}(\omega)\}$ of G from each element of Ω. The Kronecker product $\{U_g(\sigma)\otimes U_g(\tau), \mathfrak{H}(\sigma)\otimes\mathfrak{H}(\tau)\}$, in which σ, τ are elements of Ω, is decomposed into irreducible components as follows.