135. Notes on (m, n)-Ideals. II

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The first part of this paper is [2].

Now we give a characterization of groups by means of (m, n)-ideals.

Theorem 4. *A semigroup is a group if and only if it contains no proper (m, n)-ideal, where m, n are arbitrary positive integers.*

Proof. It is evident, that a group contains no proper (m, n)-ideal. Conversely, let us suppose, that the semigroup S contains no proper (m, n)-ideal. Let a be an arbitrary element of S. Then by Corollary of Theorem 2 the products aS and Sa are (m, n)-ideals of S. Hence it follows that $aS=S=Sa$. This means that for every a and b of S there exist solutions x and y in S of the equations

$$ax=b \quad \text{and} \quad ya=b,$$

that is, S is a group.

Corollary. *A semigroup is a group if and only if it contains no proper bi-ideal.*

This is the $m=n=1$ case of Theorem 4, and it is known, see [1], p. 84. (The bi-ideal is same as $(1, 1)$-ideal.)

Theorem 5. *Let m, n are arbitrary positive integers, let S be a semigroup, A be an $(m, 0)$-ideal, B a $(0, n)$-ideal of S, and suppose, that $AB=BA$. Then the product AB is an (m, n)-ideal of S.*

Proof. The suppositions of the theorem imply

$$(AB)(AB)=A^2B^2 \subseteq AB,$$

that is, the product AB is a subsemigroup of S. On the other hand

$$(AB)^m S(AB)^n = A^m(B^m SA^n)B^n \subseteq (A^m S)B^n \subseteq AB,$$

i.e. the product AB is an (m, n)-ideal of S.

In the particular case of $m=n=1$, the condition $AB=BA$ is superfluous, that is, we have the following result.

Theorem 6. *Let S be an arbitrary semigroup. If L is a left ideal and R is a right ideal of S, then the product RL is a bi-ideal of S.*

Proof. Since

$$(RL)(RL) \subseteq RL,$$

the product RL is a subsemigroup of S. On the other hand

$$(RL)S(RL) \subseteq RSL \subseteq RL,$$

that is, the product RL is a bi-ideal of S, as we stated.

If S is a regular semigroup, that is, $a \in aSa$ for each element a