153. *An Integral of the Denjoy Type*

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1. Introduction. In the present paper, we shall consider an integral of the Denjoy type whose indefinite integral is approximately continuous. H. W. Ellis [2] has introduced the GM-integral descriptively. Defining our integral we use his method, which is essentially based on the procedure introduced by S. Saks [3] and W. L. C. Sargent [4]. It will be proved that our integral is more general than Burkill's approximately continuous Perron integral [1].

2. A finite function $f(x)$ is said to be $\underline{AC}$ on a set E if to each positive number ε, there exists a number $\delta>0$ such that

$$\sum\{f(b_k)-f(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with end points on E and such that $\sum(b_k-a_k)<\delta$. There is a corresponding definition $\overline{AC}$ on E. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is $\underline{AC}$ then $f(x)$ is termed $\underline{ACG}$ on E. If the sets E_k are assumed to be closed, then $f(x)$ is said to be $(\underline{ACG})$ on E. Similarly we can define $\overline{ACG}$ and $(\overline{ACG})$ on E. A function is said to be (ACG) on E if it is both $(\underline{ACG})$ and $(\overline{ACG})$ on E.

Lemma 1. *If $F(x)$ is $\underline{AC}$ and $AD\ F(x)\geqq 0$ almost everywhere on $[a, b]$ then $F(x)$ is non-decreasing on $[a, b]$.*

Proof. Since $F(x)$ is $\underline{AC}$ on $[a, b]$, for a given $\varepsilon>0$ we can find $\delta>0$ such that

$$\sum\{F(b_k)-F(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with $\sum(b_k-a_k)<\delta$.

If we put $E=\{x: AD\ F(x)\geqq 0\}$ then $|E|=b-a$. For any $x\in E$ there exists a positive sequence h_k such that

$$\frac{F(x+h_k)-F(x)}{h_k}>-\varepsilon,\ (k=1, 2, \cdots)$$

and $h_k\to 0$. Let M be the family of the sets of closed intervals $[x, x+h_k]$ $(k=1, 2, \cdots)$ for all $x\in E$, then E is covered by M in the sense of Vitali. Hence we can select a finite sequence of non-overlapping intervals in M

$$[x_1, x_1'],\ [x_2, x_2'],\ \cdots,\ [x_m, x_m']$$

such that