2. A Note on General Connections

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Let M be an n-dimensional differentiable manifold and γ be a general connection on M. In terms of local coordinates u^i of M, γ can be written as:

(1) $$\gamma=\partial u_j\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h),^{1)}$$

where $\partial u_j=\partial/\partial u^j$ and d^2u^i is the differential of order 2 of u^i. As easily shown, P_i^j are the components of a tensor field of type (1, 1), which we denote by

(2) $$P=\partial u_j\otimes P_i^j du^i=\lambda(\gamma).$$

For any tensor field Q of type (1, 1) with local components Q_i^j and a general connection γ, we can define two general connections as

(3) $$Q\gamma=\partial u_k Q_j^k\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h)^{2)}$$

and

(4) $$\gamma Q=\partial u_j\otimes(P_k^j d(Q_i^k du^i)+\Gamma_{kh}^j(Q_i^k du^i)\otimes du^h).$$

On these multiplications of general connections and tensor fields of type (1, 1), we have

Lemma 1. *The products of general connections and tensor fields of type* (1, 1) *defined by* (3) *and* (4) *satisfy the associative law with respect to the multiplications from the left and the right.*[3]

When $\lambda(\gamma)=1$, γ is an affine connection on M and when $\lambda(\gamma)=0$, γ is a tensor field of type (1, 2). In this note, we investigate the condition that for a given general connection γ on M there exist affine connections γ_1 such that $\gamma=P\gamma_1$ or $\gamma=\gamma_1 P$, where $P=\lambda(\gamma)$. The following theorem is evident.

Theorem A. *Let γ be a general connection on M and put $P=\lambda(\gamma)$. In order to exist an affine connection γ_1 such that*

(5) $$\gamma=P\gamma_1 \quad (or\ \gamma=\gamma_1 P)$$

it is necessary and sufficient that for an affine connection γ^* on M, the tensor $T=\gamma-P\gamma^*$ (or $\gamma-\gamma^*P$) is of the form $T_{ih}^j=P_l^j V_{ih}^l$ (or $V_{lh}^j P_i^l$), where T_{ih}^j and V_{ih}^j are the local components of T and a tensor of type (1, 2).

Lemma 2. *Let V be an n-dimensional vector space over the real field R and P: $V\to V$ be a linear transformation whose minimal polynomial is*

1) See [3]

2) See [7]

3) See Proposition 1.1 in [7]