1. *Positive Pseudo-resolvents and Potentials*

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1. **Introduction.** Let Ω be a set, and denote by X a Banach space of real-valued bounded functions $f(x)$ defined on Ω and normed by $\|f\|=\sup_{x\in\Omega}|f(x)|$. We assume that X is closed with respect to the lattice operations $(f\wedge g)(x)=\min(f(x), g(x))$ and $(f\vee g)(x)=\max(f(x), g(x))$. For any linear subspace Y of X, we shall denote by Y^+ the totality of functions $f\in Y$ which are $\geqq 0$ on Ω, in symbol $f\geqq 0$. We also use the notation $f^+=f\vee 0$ and $f^-=(-f)\vee 0$.

We denote by $L(X, X)$ the totality of continuous linear operators defined on X into X. A family $\{J_\lambda;\lambda>0\}$ of operators $\in L(X, X)$ is called a *pseudo-resolvent* if it satisfies the *resolvent equation*

(1) $$J_\lambda - J_\mu = (\mu-\lambda)J_\lambda J_\mu.$$

Suggested by the case of the resolvent $J_\lambda=(\lambda I-A)^{-1}$ of the infinitesimal generator A of a semi-group $\{T_t; t\geqq 0\}$ of operators $\in L(X, X)$ of class (C_0)[1)] mapping X^+ into X^+, we shall assume conditions:

(2) J_λ is *positive*, in symbol $J_\lambda\geqq 0$, that is, $f\geqq 0$ implies $J_\lambda f\geqq 0$ for all $\lambda>0$.

(3) $$\|\lambda J_\lambda\|\leqq 1 \quad \text{for all } \lambda>0.$$

Then, an element $f\in X$ is called *superharmonic* (or *subharmonic*) if $\lambda J_\lambda f\leqq f$ (or $\lambda J_\lambda f\geqq f$) for all $\lambda>0$, and an element $f\in X$ is called a *potential* if there exists a $g\in X$ such that $f=s\text{-}\lim_{\lambda\downarrow 0} J_\lambda g$, where s-lim denotes the strong limit in X, i.e., uniform limit on Ω.

We shall be concerned with the *potential operator* V defined by

(4) $Vf=s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f$ (when $s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f^+$ and $s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f^-$ both exist).

Our main results are stated in the following two theorems.

Theorem 1. Let J_λ satisfy (1) and (2). Then $V\geqq 0$ and we have:

(5) Let $f\in X^+, g\in X^+$ and $\lambda>0$, and define $V_\lambda=V+\lambda^{-1}I$. If $(V_\lambda f)(x)\leqq (Vg)(x)$ on the support (f), we must have $V_\lambda f\leqq Vg$. (*the principle of majoration*).

Theorem 2. Let J_λ satisfy (1), (2) and (3). If the *range* $R(V)$ of the potential operator V is dense in X, then $R(V_\lambda)$ is also dense in X and the *null space* $N(V)=\{f; Vf=0\}$ consists of the zero vector only. Moreover, J_λ is the resolvent of a linear operator A with dense *domain* $D(A)$ defined through the *Poisson equation* $AVf=-f$.

Remark. Two special cases of X are important for concrete

1) See, e.g., K. Yosida: Functional Analysis, Springer, to appear soon.