35. A Note on Countable-dimensional Metric Spaces

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This paper is a supplementary note to the characterization of countable-dimensional metric spaces by J. Nagata [2]. A space is *countable-dimensional* if it is the countable sum of zero-dimensional (in the sense of the covering dimension) subsets. A space is *strongly countable-dimensional* if it is the countable sum of finite dimensional closed subsets. Now Nagata has characterized these two classes of infinite dimensional metric spaces as follows:

Theorem A [2, Theorem 2.3]. *A metric space is countable-dimensional if and only if for every collection* $\{U_\alpha: \alpha<\tau\}$ *of open sets and every collection* $\{F_\alpha: \alpha<\tau\}$ *of closed sets such that* $F_\alpha\subset U_\alpha$, $\alpha<\tau$, *and such that* $\{U_\beta: \beta<\alpha\}$ *is locally finite for every* $\alpha<\tau$, *there exists a collection of open sets* V_α, $\alpha<\tau$, *satisfying*

i) $F_\alpha\subset V_\alpha\subset U_\alpha$, $\alpha<\tau$,

ii) order $(x, \mathrm{B}(\mathfrak{V}))<\infty$ *for every* $x\in X$, *where* $\mathfrak{V}=\{V_\alpha: \alpha<\tau\}$ *and* $\mathrm{B}(\mathfrak{V})=\{\mathrm{B}(V_\alpha)=\bar{V}_\alpha-V_\alpha: \alpha<\tau\}$.

Theorem B [2, Theorem 5.3]. *A metric space* X *is strongly countable-dimensional if and only if there exists a sequence* $\mathfrak{U}_1>\mathfrak{U}_2^*>\mathfrak{U}_2>\mathfrak{U}_3^*>\cdots$ *of open coverings* $\mathfrak{U}_i$ *of* X *such that*

i) *for* $x\in X$, $\{\mathrm{St}(x, \mathfrak{U}_i): i=1,2,\cdots\}$ *is a local base of* x,

ii) *for* $x\in X$, sup order $(x, \mathfrak{U}_i)<\infty$.

Our supplementary theorems to these are as follows:

Theorem 1. *A metric space* X *is countable-dimensional if and only if for every sequence of pairs of disjoint closed sets* C_1, C_1'; C_2, C_2'; $\cdots$, *there exist separating closed sets* B_i *between* C_i *and* C_i', $i=1,2,\cdots$, *such that* $\{B_i: i=1,2,\cdots\}$ *is point-finite.*

The *only if* part of this theorem is a special case of Nagata [2, Lemma 2.1].

Theorem 2. *A metric space* X *is strongly countable-dimensional if and only if there exists a sequence* $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ *of open coverings* $\mathfrak{U}_j$ *of* X *such that*

i) *for* $x\in X$, $\{\mathrm{St}(x, U_i^i): i=1,2,\cdots\}$ *is a local base of* x,

ii) *for* $x\in X$, sup order $(x, \mathfrak{U}_i)<\infty$.

To prove Theorem 2 we need the following theorem for finite dimensional spaces.

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