22. *A Limit Theorem for Sums of a Certain Kind of Random Variables*

By Saburô UCHIYAMA

Department of Mathematics, Hokkaidô University, Sapporo

(Comm. by Zyoiti SUETUNA M.J.A., Feb. 12, 1965)

Let $X=(X, \mathscr{B}, \mu)$ be a fixed probability space, i.e. a totally finite measure space X with a measure μ such that $\mu(X)=1$. We consider a sequence of random variables

$$\varphi_m^{(h)}(x) \quad (m=1, 2, \cdots; h\geqq 2)$$

on X which are defined by the conditions:

1) Let $\rho_1, \rho_2, \cdots, \rho_h$ be the set of h-th roots of unity. The functions $\varphi_p^{(h)}(x)$ with prime-number indices p assume the values $\rho_k (1\leqq k\leqq h)$ with equal probability $1/h$ and they are (stochastically) independent.

2) For general $m\geqq 1$ the functions $\varphi_m^{(h)}(x)$ are completely multiplicative with respect to m, i.e.

$$\varphi_{ij}^{(h)}(x)=\varphi_i^{(h)}(x)\varphi_j^{(h)}(x)$$

for any positive integers i, j: in particular $\varphi_1^{(h)}(x)=1$ with probability 1.

Apparently, the functions $\varphi_m^{(h)}(x)$ $(m=1, 2, \cdots)$ are not independent.

We write

$$s_n^{(h)}(x)=\sum_{m=1}^{n} \varphi_m^{(h)}(x) \qquad (n=1, 2, \cdots).$$

Our aim in this note is to prove the following

Theorem. *We have for any* $\varepsilon>0$

$$(1) \qquad \lim_{n\to\infty} \frac{s_n^{(2)}(x)}{n^{\frac{1}{2}} (\log n)^{\frac{7}{4}+\varepsilon}}=0$$

with probability 1 *and for* $h\geqq 3$

$$(2) \qquad \lim_{n\to\infty} \frac{s_n^{(h)}(x)}{n^{\frac{1}{2}} (\log n)^{\frac{3}{2}+\varepsilon}}=0$$

with probability 1.

According to P. Erdös (Some unsolved problems. Publ. Math. Inst. Hungar. Acad. Sci., vol. 6 ser. A (1961), pp. 221–254; especially, pp. 251–252), A. Wintner proved that for any $\varepsilon>0$ we have

$$\lim_{n\to\infty} \frac{s_n^{(2)}(x)}{n^{\frac{1}{2}+\varepsilon}}=0$$

with probability 1, and Erdös himself has improved this result to

$$\lim_{n\to\infty} \frac{s_n^{(2)}(x)}{n^{\frac{1}{2}} (\log n)^{c}}=0$$