21. *A Certain System of Parameters in a Local Ring*

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Let R be a (noetherian) local ring with maximal ideal $\mathfrak{M}$: denoted by $(R, \mathfrak{M})$. First we set the following

Definition. In a local ring $(R, \mathfrak{M})$ we call a system of parameters $a_1, \cdots, a_r$ of R satisfying the conditions; $a_i \notin \mathfrak{M}^2 + \sum_{j<i} a_j R\ (1 \leq i \leq r)$ a *special system of parameters* of R, where $r = \text{Alt.}\, R$[1] (altitude of R = Krull dimension of R).

In this note, by using this notion of a special system of parameters, we shall prove the following:

Theorem. *In a local ring $(R, \mathfrak{M})$ the following three conditions are equivalent to each other*:

(1) *R is a Macaulay local ring.*

(2) *If $a_1, \cdots, a_r$ is a system of parameters of R, then $hd_R \sum_{i=1}^{r} a_i R$[1] $< \infty$.*

(3) *There exists a special system of parameters $a_1, \cdots, a_r$ such that $hd_R \sum_{i=1}^{r} a_i R < \infty$.*

For the proof of the theorem we need the following lemmas.

Lemma 1. *Let $\mathfrak{A}$ be an ideal of a local ring $(R, \mathfrak{M})$ and $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be prime ideals of R. If $\mathfrak{B}$ is an ideal of R such that $\mathfrak{B} \not\subseteq \mathfrak{A}$ and $\mathfrak{B} \not\subseteq \bigcup_{j=1}^{n} \mathfrak{P}$, then $\mathfrak{B} \not\subseteq \mathfrak{A} \cup \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_n$.*

Proof. See [2, p. 70. Prop. 2].

Lemma 2. In a local ring $(R, \mathfrak{M})$ there exists a special system of parameters.

Proof. We shall show how to construct inductively such a system of parameters. It is obvious if $\text{Alt.}\, R = 0$. Let $r = \text{Alt.}\, R \geq 1$ and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the minimal prime divisors of zero. Take a_1 such that $a_1 \in \mathfrak{M}$, $a_1 \notin \mathfrak{M}^2$ and $a_1 \notin \mathfrak{P}_i\ (i=1, \cdots, n)$ by Lemma 1. Then the height of $a_1 R$ is one. After choosing $a_i, \cdots, a_t\ (t < r)$, we can take a_{t+1} in M such that $a_{t+1} \notin \mathfrak{M}^2 + \sum_{j=1}^{t} a_i R$ and $a_{t+1} \notin \mathfrak{Q}_i\ (i=1, \cdots, m)$ by Lemma 1, where $\mathfrak{Q}_j$'s are the minimal prime divisors of $\sum_{j=1}^{t} a_j R$. It is obvious that the height of $\sum_{j=1}^{t+1} a_j R$ is $t+1$. Thus, we obtain a special system of parameters $a_1, \cdots, a_r$ of R.

Remark. If $a_1, \cdots, a_r$ is a special system of parameters of R

1) Concerning notations see [3].