48. On the Ranges of the Increasing Mappings

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Let E be a real Banach space, G be an open subset and $\bar{G}$ be the closure of G. In [3] (cf. [4] and [5]), we gave the following definitions:

A mapping $f: \bar{G} \to E$ *is said to be* (δ_0)*-increasing at* $a \in G$ *if* f *satisfies the following two conditions:*

1°. $\|x\| < \delta_0$ *implies* $a + x \in G$;

2°. $f(a+x) - f(a) \neq \alpha x$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta_0$.

A mapping $f: \bar{G} \to E$ *is said to be* $(\varepsilon_0, \delta_0)$*-uniformly increasing at* $a \in G$ *if* f *satisfies the following conditions:*

1°. $\|x\| < \delta_0$ *implies* $a + x \in G$;

3°. $\|f(a+x) - f(a) - \alpha x\| \geqq \varepsilon_0 \|x\|$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta_0$.

It is evident that, if a mapping $f: \bar{G} \to E$ is $(\varepsilon_0, \delta_0)$-uniformly increasing at a, then f is (δ_0)-increasing at a.

The following two facts immediately follow from the above definitions.

Theorem 1. *If a mapping* $f: E \to E$ *is* (∞)*-increasing at every point of* E, *then* f *is one-to-one.*

Theorem 2. *If a mapping* $f: E \to E$ *is* (ε_0, ∞)*-uniformly increasing at every point of* E, *then, for any non-positive number* α, *the range of* $f(x) - \alpha x$ *is closed.*

A mapping $f: \bar{G} \to E$ is said to be *a completely continuous vector field on* $\bar{G}$ if f is continuous on $\bar{G}$ and the image $F(\bar{G})$ by the mapping $F(x) = x - f(x)$ is contained in a compact set. We shall say that f is a completely continuous vector field *on* E if it is a completely continuous vector field on any closed ball $\bar{B}(r) = \{x \in E \mid \|x\| \leqq r\}$.

Then, we can prove the following

Theorem 3. *Let* $f: E \to E$ *be a mapping. Suppose that*

4°. f *is* (ε_0, ∞)*-uniformly increasing at every point of* E;

5°. f *is a completely continuous vector field on* E.

Then, the mapping f *is onto, one-to-one and bicontinuous.*

Proof. Theorem 1 and the condition 4° imply that f is one-to-one. Theorem 2 and the condition 4° imply that $f(E)$ is closed. We have only to prove that $f(E)$ is open.

Assume that $y_0 \in f(E)$, namely, $y_0 = f(x_0)$ for some $x_0 \in E$. There

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