65. On Automorphisms of Abelian von Neumann Algebras

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1. Throughout this note, we shall use the terminology due to J. Dixmier [2] without further explanations.

Following after H. A. Dye [3], we shall introduce some fundamental definitions on automorphisms of an abelian von Neumann algebra $\mathcal{A}$ with the faithful normal trace ϕ normalized by $\phi(1)=1$. A projection P in $\mathcal{A}$ is said to be *absolutely fixed* under an automorphism g of $\mathcal{A}$ if $Q^g=Q$ for each $Q\leqq P$. For the given two automorphisms g and h of $\mathcal{A}$, we shall denote by $F(g, h)$ the maximal projection in $\mathcal{A}$ which is absolutely fixed under gh^{-1}.

Let G be a group of ϕ-preserving automorphisms of $\mathcal{A}$;

$$\phi(A^g)=\phi(A) \quad \text{for each } A\in\mathcal{A} \text{ and } g\in G.$$

If $F(g, 1)=0$ for each $g\neq 1$ in G, then G is called *freely acting*. If α is an automorphism of $\mathcal{A}$, we say that α *depends* on G if $\text{l.u.b}_{g\in G} F(\alpha, g)=1$. We shall denote by $[G]$ the collection of all automorphisms of $\mathcal{A}$ which preserve ϕ and depend on G. We shall call $[G]$ the *full group* determined by G.

In this paper, we shall give a characterization of dependence of an automorphism with respect to the given group G in terms of the crossed product of an abelian von Neumann algebra $\mathcal{A}$.

2. At first we shall review briefly the concept of the crossed product of an abelian von Neumann algebra by an enumerable freely acting group G of ϕ-preservin automorphisms of $\mathcal{A}$, cf. [1], [4], and [5].

We shall denote an operator valued function defined on G by $\sum_{g\in G} g\otimes A_g$ where $A_g\in\mathcal{A}$ is the value of the function at $g\in G$. Let $\mathcal{D}$ be the set of all functions such that $A_g=0$ up to a finite subset of G. Then $\mathcal{D}$ is a linear space with the usual operations of the addition and the scalar multiplication, and becomes a $*$-algebra by the following operations:

$$\left(\sum_{g\in G} g\otimes A_g\right)\left(\sum_{h\in G} h\otimes B_h\right)=\sum_{g,h\in G} gh\otimes A_g B_h^{g^{-1}}$$

and

$$\left(\sum_{g\in G} g\otimes A_g\right)^*=\sum_{g\in G} g^{-1}\otimes A_g^{*g}.$$

For a trace ϕ in $\mathcal{A}$, we shall introduce a trace φ in $\mathcal{D}$ by

$$\varphi(g\otimes A_g)=\begin{cases}\phi(A_g) & \text{for } g=1,\\ 0 & \text{for } g\neq 1,\end{cases}$$