64. On the Conditional Expectation of a Partial Isometry in a Certain von Neumann Algebra

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1. In 1943, J. von Neumann stated in his monumental paper [3; Lemma 5.2.3] that the crossed product of an abelian von Neumann algebra with a faithful normal trace by an ergodic abelian group of trace preserving automorphisms is a continuous hyperfinite factor. He wrote that the lemma required some rather deep analysis on the decomposition of measure preserving transformations and the proof would be published in a separate paper. Since then no proof is avairable to us.

Twenty years after, H. A. Dye published an interesting paper [2] on a deep analysis of measure preserving transformations and established a theorem [2; Cor. 6.1] which implies the above cited lemma of von Neumann. It seems that Dye's success is an important advance in the recent theory of von Neumann algebras.

However, a slight defect is admitted in a part of Dye's proof. His proof depends principally upon [2; Lemma 6.1]: Let $\mathcal{A}$ be a regular maximal abelian self-adjoint subalgebra of a von Neumann algebra $\mathcal{B}$ of finite type with the faithful normal trace, and let C be an intermediate von Neumann subalgebra between $\mathcal{A}$ and $\mathcal{B}$. If U is a unitary operator of $\mathcal{B}$ which preserves $\mathcal{A}$ in the sense that $U\mathcal{A}U^*\subseteqq\mathcal{A}$, then the conditional expectation $U^E=E[U|C]$ of U conditioned by C in the sense of H. Umegaki [4] is a partial isometry of $\mathcal{B}$. To prove this, he stated at [2; (6.4)] that

$$(*)\qquad UAU^*=VAV^*,$$

for all $A\in\mathcal{A}$, where V is a partial isometry belonging to the polar decomposition of U^E, that is,

$$U^E=E[U|C]=V[U^{E*}U^E]^{\frac{1}{2}}.$$

Unfortunately, (*) is not true in general.

But, the essential part of the proof of Dye will be salvaged by the following

THEOREM. *If $\mathcal{A}$ is a maximal abelian subalgebra of a von Neumann algebra $\mathcal{B}$ with a faithful finite normal trace, if C is an intermediate von Neumann subalgebra between $\mathcal{A}$ and $\mathcal{B}$, and if U is a partial isometry of $\mathcal{B}$ preserving $\mathcal{A}$ in the sense that*