# 63. Contraction of the Group of Diffeomorphisms of $R^n$

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In this note, we show that the group of all diffeomorphisms of class $C^r(1\leq r\leq\infty)$ of $R^n$ is contractible to $O(n)$ under the $C^{r'}$-topology. $(1\leq r'\leq r)$.

*The group of diffeomorphisms.* Let $f: R^n\to R^n$ be a diffeomorphism of class $C^r$ and set

$$f(x)=(f_1(x),\cdots,f_n(x)) \qquad (x\in R^n),$$

where each $f_i(x)$ is a $C^r$-function on $R^n$. Furthermore, we set

$$|f(x)|=\sqrt{\sum_i |f_i(x)|^2},$$

$$D^pf(x)=(D^pf_1(x),\cdots,D^pf_n(x)),\ D^p=\frac{\partial^{|p|}}{\partial x^{i_1}\cdots\partial x^{i_n}},$$

$$p=(i_1,\cdots,i_n),\ |p|=i_1+\cdots+i_n,$$

$$J(f)(x)=\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}.$$

The set of all $C^r$-diffeomorphisms of $R^n$ forms a group. For any $\varepsilon>0$ and an compact set $K$ of $R^n$, consider the following subset of this group:

$$U(f,K,\varepsilon)=\{g\,|\,|f(x)-g(x)|<\varepsilon,\ |D^pf(x)-D^pg(x)|<\varepsilon,\ |p|\leq r',\ x\in K\},$$

where $i\leq r'\leq r$.

Taking these $U(f,K,\varepsilon)$ as the open basis, the group of all $C^r$-diffeomorphisms becomes a topological group. (Cerf [1], 1, 4, 2. Proposition 2, 4°. (p. 287)). We denote this group by $H^{r,r'}(n)$ and denote the subgroup of $H^{r,r'}(n)$ formed by those diffeomorphisms fixing the origin by $H_0^{r,r'}(n)$. The contraction $\rho: H^{r,r'}(n)\times I\to H^{r,r'}(n)$ defined by

$$\rho(f,t)=(f_1(x)-tf_1(0),\cdots,f_n(x)-tf_n(0)),$$

shows that $H_0^{r,r'}(n)$ is a strong deformation retract of $H^{r,r'}(n)$. Hence in the remainder, we consider the group $H_0^{r,r'}(n)$.

*Homomorphisms $J_0$ and $\iota$.* Set

$$J_0(f)=J(f)(0),\ f\in H_0^{r,r'}(n),$$

$$\iota(a_{ij})=\Big(\sum_i a_{i1}x_i,\cdots,\sum_i a_{in}x_i\Big),\ (a_{ij})\in GL(n,R).$$

Then, for

$$U((a_{ij}),\varepsilon)=\Big\{(b_{ij})\,|\,\sqrt{\sum_{ij}(a_{ij}-b_{ij})^2}<\varepsilon\Big\},$$

we have

$$J_0(U(f,K,\varepsilon/n))\subset U(J_0(f),\varepsilon),\quad \text{if } 0\in K,$$