116. Tauberian Theorems for Cesàro Sums. I

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1. Introduction. Let s_n and S_n^α be partial sum and Cesàro sum of order $\alpha\,(\alpha>-1)$, of a series $\sum_{n=0}^{\infty} a_n$ respectively. It is well-known that $S_n^\alpha=\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu$, where $A_\nu^{\alpha-1}$ is a coefficient of x^ν in $(1-x)^{-\alpha}$ $(|x|<1)$, and let σ_n^α be the Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$, that is, $\sigma_n^\alpha=S_n^\alpha/A_n^\alpha$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be (C,α)-summable $(\alpha>-1)$ to s if $\sigma_n^\alpha\to s$ as $n\to\infty$.

A. L. Dixon and W. L. Ferrar [2] proved the following theorem.

THEOREM A. If $W(x)$ and $V(x)$ are positive increasing functions of $x>0$, and $S_n^\delta=o(W(n))$, $(\delta>0)$, $s_n=S_n^0=O(V(n))$, then

$$S_n^\gamma=o[(V(n))^{1-\gamma/\delta}\,(W(n))^{\gamma/\delta}],\ (0<\gamma<\delta).$$

This theorem was generalized by K. Kanno [4] making use of the L. S. Bosanquet's method [1], as follows.

THEOREM B. Let $W(x)$ and $V(x)$ be positive functions of $x>0$ and satisfy the following conditions:

$$(1.1)\quad \begin{cases} \text{(i) there exists a real number } \beta>0 \text{ such that } n^\beta V(n) \text{ is non-decreasing;} \\ \text{(ii) } W(n) \text{ is non-decreasing;} \\ \text{(iii) } W(n)=O(V(n)) \text{ as } n\to\infty. \end{cases}$$

And if

$$(1.2)\qquad s_n=O(n^\beta V(n))$$

and

$$(1.3)\qquad S_n^\delta=o(n^\alpha W(n)) \text{ as } n\to\infty,$$

where $\delta+\beta\geqq\alpha>-1$, then

$$(1.4)\qquad S_n^\gamma=o[n^{(\delta-\gamma)\beta/\delta+\alpha\gamma/\delta}\,(V(n))^{1-\gamma/\delta}\,(W(n))^{\gamma/\delta}],\ (0<\gamma<\delta).$$

M. S. Rangachari [5] tried to generalize Theorem A. However, it seems to me that his proposition is lacking in one condition. Our attempt here is to add the condition (1.5) (iv) to his. This is the following Theorem 1.

THEOREM 1. Let $W(x)$ and $V(x)$ be positive functions of $x>0$, such that