113. On a Theorem of G. Pólya

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Let a_n $(n=0, 1, 2, \cdots)$ be a sequence of algebraic integers. In 1920 G. Pólya [2] proved that if $\sum_{n=0}^{\infty} na_n z^n$ is a rational function of z, then so is $\sum_{n=0}^{\infty} a_n z^n$. This result has recently been generalized by D. G. Cantor [1], who showed that if $f(x)$ is a non-zero polynomial in x with arbitrary complex coefficients and if $\sum_{n=0}^{\infty} f(n) a_n z^n$ is a rational function, then $\sum_{n=0}^{\infty} a_n z^n$ is again a rational function. In the present note we shall prove the following theorem which is a generalization of the above result due to Pólya in another direction:

Theorem. *Let a_n $(n=0, 1, 2, \cdots)$ be a sequence of numbers belonging to a fixed module over the ring of rational integers with a finite basis in the field of complex numbers. If $\sum_{n=0}^{\infty} na_n z^n$ is a rational function, then so is also $\sum_{n=0}^{\infty} a_n z^n$.*

It is quite easy to see that if the a_n are algebraic integers and if $\sum_{n=0}^{\infty} na_n z^n$ is a rational function, then there exists a finite algebraic extension k of the field of rational numbers such that the ring $\mathfrak{o}(k)$ of algebraic integers of k contains all of the a_n; and, as is well known, the ring $\mathfrak{o}(k)$ has as a module a finite basis over the ring of rational integers.

1. Lemmas. Let K_1 be an arbitrary field of characteristic 0 and K_2 a field containing K_1. We require the following two lemmas which are substantially proved in [2; pp. 4–5].

Lemma 1. *Let $A(z)$ be a non-zero polynomial of $K_1[z]$ and write*

$$A(z)=(P_1(z))^{e_1} \cdots (P_r(z))^{e_r},$$

where $P_1(z), \cdots, P_r(z)$ are distinct irreducible polynomials in $K_1[z]$ and $e_1, \cdots, e_r$ are positive integers. If $B(z)$ is a polynomial of $K_2[z]$, then we have

$$\frac{B(z)}{A(z)}=\sum_{j=1}^{r} \frac{B_j(z)}{(P_j(z))^{e_j}}$$

for some polynomials $B_1(z), \cdots, B_r(z)$ of $K_2[z]$.

Proof. Clear.

Lemma 2. *Let $P(z)$ be an irreducible polynomial of $K_1[z]$ and $Q(z)$ be a polynomial of $K_2[z]$. Let e be a positive integer. Then there exist a rational function $\phi(z)$ of $K_2(z)$ and a polynomial $R(z)$ of $K_2[z]$ with $\deg R(z)<\deg P(z)$ such that*