112. *Boundary Convergence of Blaschke Products in the Unit-Circle. II*

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1. **Introduction.** We denote by $B(z)$ a Blaschke product in the unit circle:

$$B(z)=\prod_{n=1}^{+\infty} b(z, a_n)=\prod_{n=1}^{+\infty}\{1+c(z, a_n)\},$$

where $b(z, a)=\bar{a}/|a|\cdot(a-z)/(1-\bar{a}z)$, $0<|a_n|<1$, $S=\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. For the sake of convenience, we make here a list of notations, which are used often in the sequel:

[1] $d(z, a)=(1-|a|^2)/(\bar{a}z-1)$.

[2] $c(z, a)=(1-|a|)/|a|+1/|a|\cdot d(z, a)$.

[3] $b(z, a)=1/|a|\cdot(1+d(z, a))$
$=1+c(z, a)$.

[4] $\theta_n=\arg b(1, a_n)$, $|\theta_n|\leqq\pi$.

[5] $r_n=|d(1, a_n)|=(1-|a_n|^2)/|1-a_n|$,
$R_n=(1-|a_n|)/|1-a_n|$.

[6] $\varphi_n=\arg d(1, a_n)=\arg b_n$, where $a_n=1+b_n$, $\pi/2<|\varphi_n|\leqq\pi$.

The object of this note is to establish some new theorems on boundary convergence of $B(z)$. Our main theorems read as follows:

Theorem 1.

(1.1) $\sum_{n=1}^{+\infty} R_n<+\infty$, *if and only if the following conditions hold simultaneously*:

$$(1.2)\qquad \begin{aligned}&(1)\quad \sum_{n=1}^{+\infty}|\theta_n|<+\infty,\\ &(2)\quad \lim_{n\to+\infty} R_n=0.\end{aligned}$$

Remark 1. (1) By the inequalities:

$$|c(1,a_n)|-(1-|a_n|)/|a_n|\leqq R_n\cdot(1+1/|a_n|)\leqq|c(1,a_n)|+(1-|a_n|)/|a_n|$$

$\sum_{n=1}^{+\infty}|c(1, a_n)|<+\infty$ *is equivalent to* $\sum_{n=1}^{+\infty} R_n<+\infty$ ([4] p. 67).

(2) In connection with Theorem 1, the following theorem due to O. Frostman ([2] p. 2) is very interesting; *The necessary and sufficient condition that $B(z)$ and all its partial products have the radial limit of modulus one at $z=1$ is that* $\sum_{n=1}^{+\infty} R_n<+\infty$.

Theorem 2 and 3 give the necessary and sufficient conditions for $\sum_{n=1}^{+\infty} c(1, a_n)$ to be convergent.