§1. Introduction. In [2] R. Salem and A. Zygmund proved the

Theorem. Let \( S_N(t) = \sum_{k=1}^{N} a_k \cos 2\pi n_k(t + \alpha_k) \) and \( A_N = (2^{-\frac{1}{2}} \sum_{k=1}^{N} a_k^2)^{1/2} \),
where \( \{n_k\} \) is a sequence of positive integers satisfying
\[
(1.1) \quad n_{k+1} > n_k (1 + c), \text{ for some } c > 0,
\]
and \( \{a_k\} \) an arbitrary sequence of real numbers for which
\[
A_N \to +\infty, \quad \text{and} \quad |a_N| = o(A_N), \quad \text{as } N \to +\infty.
\]
Then we have, for any set \( E \subseteq [0, 1] \) of positive measure and \( x \),
\[
(1.2) \quad \lim_{N \to \infty} \left| \int_{[t; t+1]} S_N(t) \, dt \right| = \frac{2}{\pi} \int_{-\infty}^{ \frac{\pi}{2} } \exp(-u^2/2) \, du.\quad \text{(*)}
\]

Recently, it is proved that the lacunarity condition (1.1) can be relaxed in some cases (c.f. [1] and [4]). But in [1] it is pointed out that to every constant \( c > 0 \), there exists a sequence \( \{n_k\} \) for which \( n_{k+1} > n_k (1 + ck^{-1/2}) \) but (1.2) is not true for \( a_k = 1 \) and \( E = [0, 1] \).

The purpose of the present note is to prove the following

Theorem. Let \( S_N(t) = \sum_{k=1}^{N} a_k \cos 2\pi n_k(t + \alpha_k) \) and \( A_N = (2^{-\frac{1}{2}} \sum_{k=1}^{N} a_k^2)^{1/2} \),
where \( \{n_k\} \) is a sequence of positive integers satisfying
\[
(1.3) \quad n_{k+1} > n_k (1 + ck^{-\alpha}), \text{ for some } c > 0 \text{ and } 0 \leq \alpha \leq 1/2,
\]
and \( \{a_k\} \) an arbitrary sequence of real numbers for which
\[
(1.4) \quad A_N \to +\infty, \quad \text{and} \quad |a_N| = o(A_N N^{-\alpha}), \quad \text{as } N \to +\infty.
\]
Then (1.2) holds, for any set \( E \subseteq [0, 1] \) of positive measure.

From the above theorem we can easily obtain the

Corollary. Under the conditions (1.3) and (1.4), we have
\[
(1.5) \quad \limsup_{N \to \infty} \sum_{k=1}^{N} a_k \cos 2\pi n_k(t + \alpha_k) = +\infty, \quad \text{a.e. in } t.
\]

For the proof of our theorem we use the following lemma which is a special case of Theorem 1 in [3].

Lemma 1. Let \( S_N(t) = \sum_{k=1}^{N} a_k \cos 2\pi k(t + \alpha_k) \) and \( A_N = (2^{-\frac{1}{2}} \sum_{k=1}^{N} a_k^2)^{1/2} \),
then we put \( A_k(t) = S_{k+1}(t) - S_k(t) \) and \( B_N = A_N^{k+1} \). Suppose if
\[
B_N \to +\infty, \quad \text{and} \quad \sup \left| A_N(t) \right| = o(B_N), \quad \text{as } N \to +\infty,
\]
and
\[
\lim_{N \to \infty} \int_{0}^{1} B_N^{-2} \sum_{k=1}^{N} \left[ \frac{A_k(t)^2 + 2A_k(t)B_k(t)}{A_{k+1}(t)} - 2 \right] \, dt = 0,
\]
then (1.2) holds, for any set \( E \subseteq [0, 1] \) of positive measure.

\(^*) \quad |E| \text{ denotes the Lebesgue measure of the measurable set } E.\)