152. On a J. v. Neumann's Theorem

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J. v. Neumann proved the following theorem in his paper "Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren".

In Hilbert space with the weak topology the first axiom of countability of Hausdorff does not hold.

We will give an alternative proof for this theorem.

Lemma. If $\mathfrak{B}$ is a vector basis (Hamel basis) of a Hilbert space $\mathfrak{H}$, then the cardinal number of $\mathfrak{B}$ is greater than $\aleph_0$.

Suppose that the cardinal number of $\mathfrak{B}$ is $\aleph_0$, then all elements of $\mathfrak{B}$ can be written in the following way

$$\mathfrak{B}=\{\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots\}.$$

Let us consider the sets F_n $(n=1, 2, 3, \cdots)$ consisting of all linear combinations of $\varphi_1, \varphi_2, \cdots, \varphi_n$, then every set F_n is nowhere dense. Let $\mathfrak{L}(F_n)$ be the linear hull of set F_n, then, since $UFn \supseteq \mathfrak{B}$,

$$\therefore \quad \mathfrak{L}(UF_n)\supseteq\mathfrak{L}(\mathfrak{B}),$$

on the other hand $\mathfrak{L}(UF_n) = UF_n$, $\mathfrak{L}(\mathfrak{B})=\mathfrak{H}$,

$$\therefore \quad \mathfrak{H}=UF_n.$$

This contradicts the fact that in Hilbert space the Baire category theorem holds.

Proof of Theorem. Let $\mathfrak{B}$ be the vector basis (Hamel basis) of the Hilbert space $\mathfrak{H}$, let us consider the following family of sets as a system of neighborhoods for any $f_0\in\mathfrak{H}$, and denote it by $\mathfrak{V}_1(f_0)$:

$$\mathfrak{V}_1(f_0)=\left\{u_2\left(f_0; \psi_1, \psi_2, \cdots, \psi_t, \frac{1}{n}\right)\right.$$

$$\left.\psi_1, \psi_2, \cdots, \psi_t\in\mathfrak{B}, n=1, 2, \cdots, t=1, 2, \cdots\right\},$$

where the sets $u_2\left(f_0; \psi_1, \cdots, \psi_t, \frac{1}{n}\right)$ consist of all elements $f\in\mathfrak{H}$ such that

$$|(f-f_0, \psi_1)|<\frac{1}{n}, |(f-f_0, \psi_2)|<\frac{1}{n}, \cdots$$

$$\cdots, |(f-f_0, \psi_t)|<\frac{1}{n}.$$

In Hilbert space $\mathfrak{H}$ with the weak topology the system $\mathfrak{V}(f_0)$ of neighborhoods for any element $f_0\in\mathfrak{H}$ consists of all sets $u_2(f_0; \varphi_1, \cdots, \varphi_s, \varepsilon)$, where $\varphi_1, \varphi_2, \cdots, \varphi_s$ belong to $\mathfrak{H}$, $s=1, 2, \cdots$ and $0<\varepsilon<\infty$.